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STURM - LIOUVILLE SYSTEMS AND NONSELFADJOINT OPERATORS, PRESENTED AS COUPLINGS OF DISSIPATIVE AND ANTIDISSIPATIVE OPERATORS WITH REAL ABSOLUTELY CONTINUOUS SPECTRA*

GALINA S. BORISOVA

ABSTRACT: *This paper is a continuation of the considerations of the paper [1] and it presents the connection between Sturm-Liouville systems and Livšic operator colligations theory. An usefull representation of solutions of Sturm - Liouville systems is obtained using the resolvent of operators from a large class of nonselfadjoint nondissipative operators, presented as couplings of dissipative and antidissipative operators with real spectra. A connection between Sturm-Liouville systems and the inner state of the corresponding open system of operators from the considered class is presented.*

KEYWORDS: *Sturm-Liouville system, Dissipative operator, Operator colligation, Triangular model, Coupling, Open system.*

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1 Preliminary results

In this paper we present an usefull representation of solutions of the Sturm-Liouville system

$$\frac{d^2}{dx^2}f(x,\lambda) - q(x)f(x,\lambda) + \lambda f(x,\lambda) = 0 \tag{1}$$

on the half axis $x \geq 0$ (f is $\mathbb{C}^r$-valued vector function, $q(x)$ is $r \times r$ matrix function on $[0,\infty)$, and λ is the spectral parameter ($\lambda \in \mathbb{C}$)), connected with the resolvents of the operators from the large class of

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nonselfadjoint nondissipative operators, presented as couplings of dissipative and antidissipative operators with finite dimensional iaginary parts and real nonzero spectra. The obtained results present the connection between Sturm-Liouville systems and the nonselfadjoint operator colligations Livšic theory. These results can be applied in the connection between commuting nonselfadjoint operators and solitonic combinations, established by M.S. Livšic and Y. Avishai in [9], considered and developed by G.S. Borisova and K.P. Kirchev in [4], and by G.S. Borisova in [2].

A similar problem has been considered in [5] by I. Gohberg, M.A. Kaashoek, A.L. Sakhnovich for a nonselfadjoint operator A with finite dimensional imaginary part and zero spectrum in a Hilbert space H and in a special case (when A satisfies additional conditions) the operator A has the representation $iA = P_1AP_1 + P_2AP_2$ (P_1 and P_2 are orthoprojectors in H, $H = P_1H \oplus P_2H$), P_1AP_1 is a dissipative operator and P_2AP_2 is an antidissipative operator, i.e. A is a sum of a dissipative operator and antidissipative one with zero spectra). In the paper [1] the case with an operator, presented as a coupling of dissipative and antidissipative operators with zero spectra has been considered.

In this paper, we consider the case of a nonselfadjoint operator A with finite dimensional imaginary part in a Hilbert space, presented as a coupling of a dissipative operator P_1AP_1 and an antidissipative operator P_2AP_2 with real nonzero spectra, i.e. A has the form

$$A = P_1AP_1 + P_2AP_2 + P_1AP_2.$$

The natural consideration of this larger class of nondissipative nonselfadjoint operators follows from the system-theoretic significance of the colligation which is connected with the multiplication theorem of the corresponding characteristic functions. The asymptotic behaviour of the corresponding nondissipative curves $e^{itA}f$ as $t \to \pm\infty$ ($f \in H$) (obtained in [7], [8]) ensures the existence and the explicit form of the limits $\lim_{t\to\pm\infty}(e^{itA}f, e^{itA}f)$. This is the reason and the importance for consideration of the class of couplings of dissipative and antidissipative operators.

We consider the triangular model of couplings of dissipative and antidissipative operators with real spectra, introdused by the author in [3], and studied in [7].

Now we will introduce some denotations and preliminary results which we will use essentially. Let L be selfadjoint $m \times m$ matrix with $\det L \neq 0$, let r be the number of positive eigenvalues and $m-r$ be the number of negative eigenvalues of L. Without loss of generality we can assume that L has the form $L = J_1 - J_2 + S + S^*$ or

$$L = \begin{pmatrix} I_r & \widehat{S}^* \\ \widehat{S} & -I_{m-r} \end{pmatrix}, \quad J_1 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$
$$J_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ \widehat{S} & 0 \end{pmatrix}.$$

The model

$$\begin{aligned} Af(x) = \alpha(x)f(x) - i\int_0^x f(w)\Pi(w)dwS^*\Pi^*(x) + \\ +i\int_x^l f(w)\Pi(w)dwS\Pi^*(x) + i\int_0^x f(w)\Pi(w)dwL\Pi^*(x) \end{aligned} \tag{2}$$

(see [3], [7]) is the triangular model of a coupling of dissipative and antidissipative operators with real spectra. Here $\alpha(x)$ is a bounded non-decreasing function on a finite interval $[0,l]$ which is continuous at o and continuous from the left on $[0,l]$, $\Pi(x)$ is a measurable $n \times m$ ($1 \leq n \leq m$) matrix function on $[0,l]$, whose rows are linearly independent at each point of a set of a positive measure, and satisfying the condition

$$J_1\Pi^*(x)\Pi(x) = \Pi^*(x)\Pi(x)J_1. \tag{3}$$

Let us consider the case when $n = r$ and $m - r > r$. The cases when $m - r < r$ and $m - r = r$ can be considered analogously.

Let the selfadjoint potential $q(x)$ ($r \times r$ matrix function) be integrable on $[0, l]$. Let also the matrix function $P(x)$ have the form

$$P(x) = \begin{pmatrix} 0 & i\widetilde{G} \\ -iRq(x) & 0 \end{pmatrix}, \tag{4}$$

where $R = \widehat{S}\widehat{S}^*\widehat{S}$ and $\widetilde{G}$ is $r \times m - r$ matrix satisfying the relation $\widetilde{G}R = I_r$ (as multiplying by matrices).

Let $m \times m$ matrix function $\widehat{V}(x)$ be the solution of the equation

$$\begin{cases} \frac{d\widehat{V}(x)}{dx} = P(x)\widehat{V}(x) \\ \widehat{V}(0) = I_m \end{cases} \qquad (0 \le x \le l) \tag{5}$$

In terms of the multiplicative integrals the solution of the equation (5) has the form $\widehat{V}(x) = \overset{\curvearrowleft}{\int\limits_0^x} e^{P(w)dw}$. Let the matrix

$$\widetilde{L} = \begin{pmatrix} 0 & R^* \\ R & 0 \end{pmatrix} = \begin{pmatrix} 0 & \widehat{S}^*\widehat{S}\widehat{S}^* \\ \widehat{S}\widehat{S}^*\widehat{S} & 0 \end{pmatrix} \tag{6}$$

satisfies the additional condition $\widetilde{L}^2 = I$. Direct calculations show that the next matrix equality is true

$$P(x)\widetilde{L} + \widetilde{L}P^*(x) = 0 \tag{7}$$

using (4), (6) and the condition $\widetilde{G}R = I_r$. Now (5), (7) and the condition $\widetilde{L}^2 = I$ imply that $\widetilde{L}P(x)\widetilde{L}^2 + \widetilde{L}^2P^*(x)\widetilde{L} = 0$ and consequently

$$\widetilde{L}P(x) + P^*(x)\widetilde{L} = 0. \tag{8}$$

Then the derivative $\frac{d}{dx}(\widehat{V}^*(x)\widetilde{L}\widehat{V}(x)) = 0$ and

$$\widehat{V}(x)\widetilde{L}\widehat{V}^*(x) = \widetilde{L} \tag{9}$$

(see [1]). But the matrix $\widetilde{L}$ always can be presented in the form $\widetilde{L} = H^*JH$ where the matrices J and H have the form

(10)

$$J = J_1 - J_2 = \begin{pmatrix} I_r & 0 \\ 0 & -I_{m-r} \end{pmatrix}, \quad H = \begin{pmatrix} \frac{1+i}{2}\widehat{S}^*\widehat{S} & \widehat{S}^* \\ \frac{1+i}{2}U^*\widehat{S}^*\widehat{S} & -iU^*\widehat{S}^* \end{pmatrix},$$

U has the form $U = X(I_r \quad 0)Y$, where X, Y are unitary matrices (i.e. $XX^* = X^*X = I_r$, $YY^* = Y^*Y = I_{m-r}$). Now from $\widetilde{L} = H^*JH$ it follows that the equality (9) takes the form $V(x)JV^*(x) = \widetilde{L}$, where we have denoted $V(x) = \widehat{V}(x)H^*$. Now (5) can be written in the form

$$(11) \qquad \begin{cases} \frac{dV(x)}{dx} = P(x)V(x) \\ V(0) = \widehat{V}(0)H^* = H^* \end{cases} \qquad (0 \le x \le l).$$

Let us now consider the triangular model $A : \mathbf{L}^2(0,l;\mathbb{C}^r) \longrightarrow \mathbf{L}^2(0,l;\mathbb{C}^r)$ from the form (2), describing the couplings of dissipative and antidissipative operators with real spectra (introduced in [3], and investigated in [7], [8]) defined by the equality (2) when $\Pi(x) = V_1(x)$ is $r \times m$ matrix function from the block representation $V(x) = \begin{pmatrix} V_1(x) \\ V_2(x) \end{pmatrix}$ of the matrix function $V(x)$ from the form $V(x) = \widehat{V}(x)H^*$ and the matrix function $V_1(x) = (I_r \quad 0)V(x)$. This implies that A has the representation

$$(12) \qquad \begin{aligned} Af(x) &= \alpha(x)f(x) - i\int_0^x f(w)V_1(w)dwS^*V_1^*(x) + \\ &+ i\int_x^l f(w)V_1(w)dwSV_1^*(x) + i\int_0^x f(w)V_1(w)dwLV_1^*(x), \end{aligned}$$

where $f \in \mathbf{L}^2(0,l;\mathbb{C}^r)$, the function $\alpha(x) : [0,l] \longrightarrow \mathbb{R}$ is a bounded nondecreasing function, whose values determine the spectrum of the operator A. Let the operator A be embedded in the colligation $X = (A, \mathbf{L}^2(0,l;\mathbb{C}^r), \Phi, \mathbb{C}^m, L)$, where the operator $\Phi : \mathbf{L}^2(0,l;\mathbb{C}^r) \longrightarrow$

$\mathbb{C}^m$ has the representation $\Phi f(w) = \int\limits_0^l f(w)V_1(w)dw$ and consequently $\Phi^* h = hV_1^*(x)$ $(h \in \mathbb{C}^m)$. This implies that the colligation condition is $\frac{A-A^*}{i} = \Phi^* L\Phi$.

We consider the operator A from the model (12) which is a coupling of dissipative and antidissipative operators with real spectra when the matrix function $\Pi(x) = V_1(x)$ satisfies the additional condition (3). Now we present the operator A in the form

$$Af(x) = \alpha(x)f(x) + Bf(x), \tag{13}$$

where the operator B with zero spectrum is presented in the form

$$\begin{gathered} Bf(x) = -i\int\limits_0^x f(w)V_1(w)dwS^*V_1^*(x) + \\ +i\int\limits_x^l f(w)V_1(w)dwSV_1^*(x) + i\int\limits_0^x f(w)V_1(w)dwLV_1^*(x) = \\ = (Gf(x))V_1^*(x). \end{gathered} \tag{14}$$

If $V_1(x)$ satisfies the condition from the form (3) then the operator B is a coupling of dissipative and antidissipative operators with zero spectra.

The operators A from (12) and B from (14) have the same imaginary parts, i.e.

$$\begin{gathered} \frac{A-A^*}{i}f(x) = \frac{B-B^*}{i}f(x) = \\ = \int\limits_0^l f(w)V_1(w)LV_1^*(x)dw = \Phi^* L\Phi f(x). \end{gathered} \tag{15}$$

The first theorem presents the connection between the Sturm-Liouville system (1) and the resolvent of the triangular model B from the (14).

Theorem 1. *([1]) Let the operator B be defined by (14) and the self-adjoint potential $q(x)$ is integrable on $[0, l]$. Then the operator function $y(x, \lambda) = (I + \lambda B)^{-1}\Phi^* h$, $\quad h \in \mathbb{C}^m$, ($\lambda \neq 0$) is a solution of the Sturm-Liouville system (1) on $[0, l]$.*

The next theorem concerns an important property of the solution $y(x,\lambda) = (I + \lambda B)^{-1}\Phi^* h$ of the Sturm-Liouville system (1) when the operator B is a coupling of a dissipative operator and an antdissipative one with zero spectra.

Theorem 2. *([1]) For the coupling B of a dissipative operator and an antdissipative operator with zero spectra from the form (14) the next relation hold*

$$\overline{\textit{span}}\ \{(I + \lambda B)^{-1}\Phi^* h, h \in \mathbb{C}^m\} = \mathbf{L}^2(0, l; \mathbb{C}^r). \tag{16}$$

2 Main results

Let the triangular model A of the couplings of the dissipative and antidissipative operators with real spectra be defined as in Section 1 by

$$Af(x) = \alpha(x)f(x) + Bf(x),$$

where the Voltera operator B is defined by (14). Let all denotations and conditions be the same as in Section 1. The main result of the paper concerns the operator function $y(x,\lambda) = (I + \lambda A)^{-1}\Phi^* h$ ($h \in \mathbb{C}^m$, $-\frac{1}{\lambda}$ belongs to the resolvent set of the operator A), which is connected with the resolvent of the operator A.

Theorem 3. *Let the operator A be defined by (12) and the selfadjoint potential $q(x)$ is integrable on $[0, l]$. Then the operator function $y(x,\lambda) = (I + \lambda A)^{-1}\Phi^* h, \quad h \in \mathbb{C}^m$, ($-\frac{1}{\lambda}$ belongs to the resolvent set of the operator A) is a solution of the Sturm-Liouville system (1) on $[0, l]$.*

Proof. Let us consider at first the formal differential operator $\mathbf{l}$ of the form

$$\mathbf{l}f = -f'' + qf. \tag{17}$$

Now we denote by $\mathcal{A}$ the set of all functions from $\mathbf{L}^2(0,l;\mathbb{C}^r)$ with an absolutely continuous derivative f' and $\mathbf{l}f \in \mathbf{L}^2(0,l;\mathbb{C}^r)$ and define the sets

$$\mathcal{D}(D) = \{f : f \in \mathcal{A}, f(0) = f'(0) = f(l) = f'(l) = 0\},$$
$$\mathcal{D}(D') = \mathcal{A}.$$

Let us define the operators D and D' on the sets $\mathcal{D}(D)$ and $\mathcal{D}(D')$ correspondingly by the relations

$$Df = \mathbf{l}f, \qquad f \in \mathcal{D}(D)$$
$$D'f = \mathbf{l}f, \qquad f \in \mathcal{A}.$$

We consider now $(I+\lambda A)^{-1}\Phi^* h$ for an arbitrary λ such that $-\frac{1}{\lambda}$ belongs to the resolvent set of the operator A, i.e. $-\frac{1}{\lambda} \notin [0,l]$. Then for $(I+\lambda A)^{-1}\Phi^* h$ (for all $h \in \mathbb{C}^m$) we have

$$(I+\lambda A)^{-1}\Phi^* h \in \mathbf{L}^2(0,l;\mathbb{C}^r) = \overline{\mathit{span}}\ \{(I+\lambda B)^{-1}\Phi^* h, h \in \mathbb{C}^m\}$$

using Theorem 2. Then there exists a sequence $\{(I+\lambda B)^{-1}\Phi^* h_k\}$ ($h_k \in \mathbb{C}^m$ such that $\lim\limits_{k\to\infty}(I+\lambda B)^{-1}\Phi^* h_k = (I+\lambda A)^{-1}\Phi^* h$ as a strong limit according to the norm in $\mathbf{L}^2(0,l;\mathbb{C}^r)$. But from the well known fact that for a formal regular operator $\mathbf{l}$ it follows that D is a closed symmetric operator with an adjoint operator $D^* = D'$ and $D'^* = D$ (see, for example, [6]). Then it follows that the operator D', defined in $\mathcal{D}(D') = \mathcal{A}$, is closed operator, because D' is closed operator as adjoint to D. Consequently, from Theorem 1 we obtain

$$\begin{gathered} D'(I+\lambda B)^{-1}\Phi^* h_k = \mathbf{l}(I+\lambda B)^{-1}\Phi^* h_k = \\ = -\tfrac{d^2}{dx^2}(I+\lambda B)^{-1}\Phi^* h_k + ((I+\lambda B)^{-1}\Phi^* h_k)q(x) = \\ = \lambda(I+\lambda B)^{-1}\Phi^* h_k. \end{gathered}$$

From the last equalities we obtain that

$$\lim_{k\to\infty} D'(I+\lambda B)^{-1}\Phi^* h_k = \lambda D'(I+\lambda A)^{-1}\Phi^* h.$$

But D' is closed operator and from the equality

$$\lim_{k\to\infty}((I+\lambda B)^{-1}\Phi^* h_k, D'(I+\lambda B)^{-1}\Phi^* h_k)) = \\ = ((I+\lambda A)^{-1}\Phi^* h, \lim_{k\to\infty} D'(I+\lambda B)^{-1}\Phi^* h_k) = \\ = ((I+\lambda A)^{-1}\Phi^* h, \lambda(I+\lambda A)^{-1}\Phi^* h)$$

it follows that

$$D'(I+\lambda A)^{-1}\Phi^* h = \lambda(I+\lambda A)^{-1}\Phi^* h.$$

This implies that

$$-\tfrac{d^2}{dx^2}(I+\lambda A)^{-1}\Phi^* h + ((I+\lambda A)^{-1}\Phi^* h)q(x) = \\ = \lambda(I+\lambda A)^{-1}\Phi^* h,$$

i.e. the vector function $(I+\lambda A)^{-1}\Phi^* h$ satisfies the Sturm-Liouville equation, which presents the relation between the resolvent of the non-selfadjoint operators with absolutely continuous real spectra (presented as couplings of dissipative and antidissipative operators with real absolutely continuous spectra). The theorem is proved. □

The next theorem shows that there exists an essential connection between Sturm-Liouville systems and the inner state of the corresponding open system of the operator A, defined by (12), in the sence of the system theory.

Theorem 4. *Let the operator A be defined by (12) and the selfadjoint potential $q(x)$ is differentiable on $[0, l]$. Let A be embedded in the operator colligation*

$$X = (A; \mathbf{L}^2(0, l; \mathbb{C}^r), \Phi, \mathbb{C}^m; L),$$

where Φ has the form $\Phi f = \int_0^l f(w)V_1(w)dw$. Then f_0 from the representation of the inner state $f(t, x)$ of the corresponding of the open *system*

$$\begin{cases} i\frac{d}{dt}\mathbf{f} + A\mathbf{f} = \Phi^* L\mathbf{u} \\ \mathbf{v} = \mathbf{u} - i\Phi\mathbf{f}. \end{cases} \tag{18}$$

is a solution of the Sturm-Liouville system in the special case of separated variables in the input, the output and the inner state of the open system $\mathbf{u}(t,x) = e^{it\mu}\mathbf{u}_0(x)$*,* $\mathbf{v}(t,x) = e^{it\mu}\mathbf{v}_0(x)$*,* $\mathbf{f}(t,x) = e^{it\mu}\mathbf{f}_0(x)$ *(*$\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{C}^m$*,* $\mathbf{f}_0 \in \mathbf{L}^2(0,l;\mathbb{C}^r)$*).*

Proof. Direct calculations show that

$$\mathbf{f}_0 = (A - \mu I)^{-1}\,\Phi^* L\mathbf{u}_0. \tag{19}$$

Then from Theorem 3 it follows that $\frac{1}{\lambda}\mathbf{f}_0 = (I + \lambda A)^{-1}\Phi^* L\mathbf{u}_0$ satisfies the Sturm-Liouville system (1), if we denote $\lambda = -\frac{1}{\mu}$. □

Finally, it is worth to mention that the obtained results, concerning the model A of the coupling (12), are important because they can be applied in the further development of the problems concerning the connection between the commuting nonselfadjoint operator theory and the soliton theory, considered in [4] and [2]. The considerations of these problems are forthcoming.

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Galina S. Borisova
Faculty of Mathematics and Informatics
Konstantin Preslavsky University of Shumen
9700 Shumen, Bulgaria
e-mail: g.borisova@shu.bg