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SPECTRAL STABILITY OF THE CNOIDAL WAVES OF THE SCHRÖDINGER SYSTEM*

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ABSTRACT: *Periodic standing waves are considered for a Schrödinger system. The existence of periodic waves of cnoidal-type as well as the stability of the such solutions are studied.*

KEYWORDS: *spectral stability, periodic waves, Schrödinger equation*

2020 Math. Subject Classification: *35B35, 35B40, 35G30*

1 Introduction

In this work we consider the nonlinear Schrödinger system

$$
\begin{cases} iu_t + u_{xx} + k_1|u|^2u + \gamma v^2\overline{u} = 0 \\ iv_t + v_{xx} + k_2|v|^2v + \gamma u^2\overline{v} = 0, \end{cases} \tag{1}
$$

where k_1, k_2, γ are positive constants, u and v are complex valued functions. This equation appears in various problems, modeling many phenomena such as nonlinear optics, Bose-Einstein condensate, etc. [1, 15, 16, 22].

Our principal aim is to study the stability of semitrivial family of periodic wave solutions

$$
(u,v) = (e^{iwt}\phi(x), 0). \tag{2}
$$

where $\phi(x)$ is a real-valued periodic function and w is a real parameter.

The problem of the nonlinear stability of solitary waves for nonlinear dispersive equations goes back to the works of Benjamin [5] and

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Bona [6] (see also [2, 23, 24]). A general approach for investigating the stability of solitary waves for nonlinear equations, having a group of symmetries, was proposed in [10]. The existence and stability of solitary wave solutions for equation (1) have been studied in [19].

In recent years, the existence of periodic waves and their stability properties have been considered in numerous papers. In the periodic case, the spectra of the linearized equation depends on the choice of function space. In the space of periodic functions, the spectrum consists of isolated eigenvalues, while in the space of bounded functions the spectrum is continuous[see [20]]. Existence of periodic traveling waves, together with their stability, have been obtained in [3, 4, 12, 13, 14, 21] for the nonlinear Schrödinger equation, modified KdV equation, complex modified KdV equation, and generalized BBM equation. In [11], stability of the periodic waves for (2) which do not oscillate around zero (dnoidal type of solutions) was considered. It is proved that semitrivial periodic waves are nonlinearly stable for $k_1 > \gamma$ and spectrally unstable for $k_1 < \gamma < 3k_1$.

In this paper, we consider the spectral stability of the periodic waves for (2) of cnoidal type. We prove that for $k_1 = \gamma$ periodic waves of cnoidal type are spectrally unstable. This is achieved by using the Hamiltonian-Krein index theory developed in [17, 18].

The paper is organized as follows. In Section 2, we construct the semitrivial periodic waves. In Section 3, we setup the linearized problem and compute the index that gives stability and instability. First we establish the required spectral properties of the matrix Hill operator. Then we calculate the Hamiltonian instability index.

2 Existence of periodic waves

In this section, we give some standard preliminary results. We now construct the explicit solitons, which we later analyze for stability. We consider the semi-trivial periodic traveling waves in the form $u =$

$e^{iwt}\phi(x),\ \ v=0$. Plugging in the system (1), we get the equation

$$(3)\qquad -w\phi+\phi''+k_1\phi^3=0.$$

Let $\phi(x)=\frac{1}{\sqrt{k_1}}\varphi(x)$. For φ one obtains the equation

$$(4)\qquad \varphi''-w\varphi+\varphi^3=0.$$

Integrating once again, we obtain

$$(5)\qquad \varphi'^2-w\varphi^2+\frac{1}{2}\varphi^4=c$$

and φ is a periodic function provided that the energy level set $H(x,y)=c$ of the Hamiltonian system $dH=0$,

$$H(x,y)=y^2-wx^2+\frac{1}{2}x^4,$$

contains an oval (a simple closed real curve free of critical points). The level set $H(x,y)=c$ contains two periodic trajectories, if $w>0$, $c\in(-\frac{1}{2}w^2,0)$ and a unique periodic trajectory if $w\in\mathbf{R}$, $c>0$. Under these conditions, the solution of (5) is determined by $H(\varphi,\varphi')=c$ and φ is periodic function.

Let $w>0$, $c<0$ and $\varphi_0,-\varphi_0$ ($\varphi_0>0$) are real roots, and $i\varphi_1,-i\varphi_1$ are complex roots of polynomial $-z^4+2wx+c$. Then

$$\varphi'^2=\frac{1}{2}(\varphi_0^2-\varphi^2)(-2w+\varphi_0^2+\varphi^2).$$

Up to translations the solution is given by

$$(6)\qquad \varphi(x)=\varphi_0 cn(\alpha x,\kappa),$$

where

$$(7)\qquad \kappa^2=\frac{\varphi_0^2}{2\varphi_0^2-2w},\ \ \alpha^2=\varphi_0^2-w=-\frac{w}{1-2\kappa^2}.$$

Since the fundamental period of $cn(x)$ is $4K(\kappa)$, then the fundamental period of periodic function φ defined by (6) is $2T=\frac{4K(\kappa)}{\alpha}$.

3 Spectral stability

In this section, we consider the spectral stability of semitrivial periodic wave with respect to perturbation with same period as φ. The result relies on an instability index count for Hamiltonian systems. We start this section with some information about the spectrum operator of self-adjoint operator and some specific quantities, which are also computable.

Let $\vec{u} = e^{iwt}(\vec{\Phi}(x) + U(t,x) + iW(t,x))$, where $\vec{\Phi} = (\frac{1}{\sqrt{k_1}}\varphi, 0)$ and $(U,W) \in \mathbb{R}^4$ are perturbation functions with fundamental period $2T$. Plugging $\vec{u}$ into the equation (1) and ignoring the nonlinear terms, we get the following linear equation

$$\frac{d}{dt}\begin{pmatrix} U \\ W \end{pmatrix} = \mathscr{J}\mathscr{L}\begin{pmatrix} U \\ W \end{pmatrix}, \tag{8}$$

where operators $\mathscr{J}$ and $\mathscr{L}$ are given by

$$\mathscr{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \mathscr{L} = \begin{pmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_3 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & L_4 \end{pmatrix},$$

where

$$\begin{aligned} L_1 &= -\partial_x^2 + (w - 3\varphi^2), \\ L_2 &= -\partial_x^2 + (w - \varphi^2), \\ L_3 &= -\partial_x^2 + (w - \tfrac{\gamma}{k_1}\varphi^2), \\ L_4 &= -\partial_x^2 + (w + \tfrac{\gamma}{k_1}\varphi^2), \end{aligned} \tag{9}$$

We will consider the case $\gamma = k_1$. In this case $L_3 = L_2$. We have the following results related with the spectrum of the operator $\mathscr{L}$.

Proposition 1. *For $\gamma = k_1$ the number of negative eigenvalues of the operator $\mathscr{L}$ is four ($n(\mathscr{L}) = 4$). The kernel of $\mathscr{L}$ is three dimensional ($\dim\ker\mathscr{L} = 3$) and spanned by vectors $\vec{\psi}_1 = (\varphi', 0, 0, 0)$, $\vec{\psi}_2 = (0, 0, \varphi, 0)$, $\vec{\psi}_3 = (0, \varphi, 0, 0)$*

Proof. Since $\mathscr{L}$ is diagonal operator, then the spectrum of $\mathscr{L}$ is $\sigma(\mathscr{L}) = \sigma(L_1) \cup \sigma(L_2) \cup \sigma(L_3) \cup \sigma(L_4)$.

It is well-known that the first five eigenvalues of $\Lambda_1 = -\partial_y^2 + 6k^2 sn^2(y,k)$, with periodic boundary conditions on $[0, 4K(k)]$ are simple. These eigenvalues and corresponding eigenfunctions are:

$$\begin{aligned}
&\nu_0 = 2 + 2k^2 - 2\sqrt{1-k^2+k^4}, && \phi_0(y) = 1 - (1 + k^2 - \sqrt{1-k^2+k^4})sn^2(y,k),\\
&\nu_1 = 1 + k^2, && \phi_1(y) = cn(y,k)dn(y,k) = sn'(y,k),\\
&\nu_2 = 1 + 4k^2, && \phi_2(y) = sn(y,k)dn(y,k) = -cn'(y,k),\\
&\nu_3 = 4 + k^2, && \phi_3(y) = sn(y,k)cn(y,k) = -k^{-2}dn'(y,k),\\
&\nu_4 = 2 + 2k^2 + 2\sqrt{1-k^2+k^4}, && \phi_4(y) = 1 - (1 + k^2 + \sqrt{1-k^2+k^4})sn^2(y,k).
\end{aligned}$$

For $\Lambda_2 = -\partial_y^2 + 2k^2 sn^2(y,k)$ the first three eigenvalues and the corresponding eigenfunctions with periodic boundary conditions on $[0, 4K(k)]$ are simple and

$$\begin{aligned}
&\varepsilon_0 = k^2, && \theta_0(y) = dn(y,k),\\
&\varepsilon_1 = 1, && \theta_1(y) = cn(y,k),\\
&\varepsilon_2 = 1 + k^2, && \theta_2(y) = sn(y,k).
\end{aligned}$$

In the case of cnoidal solution, we have

$$L_1 = \alpha^2[-\partial_y^2 + 6\kappa^2 sn^2(y,\kappa) - (1 + 4\kappa^2)] = \alpha^2[\Lambda_1 - (1 + 4\kappa^2)].$$

It follows that the first five eigenvalues of the operator L_1, equipped with periodic boundary condition on $[0, 4K(k)]$ are simple and zero is the third eigenvalue.

For L_2, we have that

$$L_2 = \alpha^2[-\partial_y^2 + 2\kappa^2 sn^2(y,\kappa) - 1] = \alpha^2[\Lambda_2 - 1].$$

From the above, we have that $n(L_1) = 2$, $n(L_2) = 1$. Moreover $\dim\ker L_1 = \dim\ker L_2 = 1$ with corresponding eigenfunctions φ' and φ ($L_1\varphi' = 0$, $L_2\varphi = 0$). Since the operator L_4 is strong positive, the number of negative eigenvalues of the operator $\mathscr{L}$ is four ($n(\mathscr{L}) = 4$) and the kernel of $\mathscr{L}$ is three dimensional and spanned by $\vec{\psi}_1 = (\varphi', 0, 0, 0)$, $\vec{\psi}_2 = (0, 0, \varphi, 0)$, $\vec{\psi}_3 = (0, \varphi, 0, 0)$. $\square$

Definition 1. *The stationary solution* $(e^{iwt}\frac{1}{\sqrt{k_1}}\varphi, 0)$ *is spectrally unstable if there exists at least one eigenvalue* λ *of the operator* $\mathscr{J}\mathscr{L}$ *with positive real part.*

First, we present the theory developed in [17, 18] for the study of spectral problem

$$\mathscr{J}\mathscr{L}u = \lambda u, \tag{10}$$

which we will be able to apply to our problem (8). We only present a corollary of the results therein, which fits our purposes. We start with some notations. We assume that $\mathscr{L}$ has a finite number of negative eigenvalues $n(\mathscr{L}) < \infty$ and $\mathscr{J} : Ker(\mathscr{L}) \to Ker(\mathscr{L})^\perp$. Let $\mathscr{L}\psi_i = 0$ and V is a matrix with elements $V_{ij} = \langle \mathscr{L}^{-1}\mathscr{J}\psi_i, \mathscr{J}\psi_j \rangle$. For a self-adjoint operator H, define the number of the negative eigenvalues

$$n(H) = \#\{\lambda \in (-\infty, 0) \cap \sigma(H)$$

The following formula is derived in [18]

$$k_r + 2k_c + 2k_- = n(\mathscr{L}) - n(V), \tag{11}$$

where k_r is the number of positive solutions λ of (10), k_c is the number of solutions of λ with nonzero real and imaginary parts, whereas k_- is the number of pairs of purely imaginary eigenvalues with negative Krein signature.

Remark: The non-negative integers k_c, k_- are even. As a consequence, if $n(\mathscr{L}) = 1$, it follows from (11) that $k_c = k_- = 0$ and

$$k_r = 1 - n(V). \tag{12}$$

Moreover, if right hand side in (11) is odd number, then $k_r \geq 1$ and we have instability.

We use the index counting theory to determine the spectral stability of the waves. We have the following result.

Theorem 1. *Let $\gamma = k_1$. Semitrivial periodic wave solution $(e^{iwt}\frac{1}{\sqrt{k_1}}\varphi, 0)$, where φ is given by (6) is spectrally unstable.*

Proof. From Proposition 1, we have $n(\mathscr{L}) = 4$, $dimKer\mathscr{L} = 3$, and $\vec{\psi}_1 = (\varphi', 0, 0, 0), \vec{\psi}_2 = (0, 0, \varphi, 0), \vec{\psi}_3 = (0, \varphi, 0, 0) \in Ker\mathscr{L}$.

The anti-selfadjointness of $\mathscr{J}$ yields $\langle \mathscr{J}\vec{\psi}_i, \vec{\psi}_i\rangle = 0, i = 1, 2, 3$. By direct computations $\langle \mathscr{J}\vec{\psi}_i, \vec{\psi}_j\rangle = 0,\ i, j = 1, 2, 3$. With this, we have verified that $\mathscr{J} : Ker(\mathscr{L}) \to Ker(\mathscr{L})^{\perp}$.

Thus, in order to determine the stability, one needs to compute the index $n(V)$. For V, we have the following matrix representation

$$V := \begin{pmatrix} \langle \mathscr{L}^{-1}\mathscr{J}\vec{\psi}_1, \mathscr{J}\vec{\psi}_1\rangle & \langle \mathscr{L}^{-1}\mathscr{J}\vec{\psi}_1, \mathscr{J}\vec{\psi}_2\rangle & \langle \mathscr{L}^{-1}\mathscr{J}\vec{\psi}_1, \mathscr{J}\vec{\psi}_3\rangle \\ \langle \mathscr{L}^{-1}\mathscr{J}\vec{\psi}_2, \mathscr{J}\vec{\psi}_1\rangle & \langle \mathscr{L}^{-1}\mathscr{J}\vec{\psi}_2, \mathscr{J}\vec{\psi}_2\rangle & \langle \mathscr{L}^{-1}\mathscr{J}\vec{\psi}_2, \mathscr{J}\vec{\psi}_3\rangle \\ \langle \mathscr{L}^{-1}\mathscr{J}\vec{\psi}_3, \mathscr{J}\vec{\psi}_1\rangle & \langle \mathscr{L}^{-1}\mathscr{J}\vec{\psi}_3, \mathscr{J}\vec{\psi}_2\rangle & \langle \mathscr{L}^{-1}\mathscr{J}\vec{\psi}_3, \mathscr{J}\vec{\psi}_3\rangle \end{pmatrix}$$

We have

$$\mathscr{J}\vec{\psi}_1 = \begin{pmatrix} 0 \\ 0 \\ -\varphi' \\ 0 \end{pmatrix}, \quad \mathscr{J}\vec{\psi}_2 = \begin{pmatrix} \varphi \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathscr{J}\vec{\psi}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\varphi \end{pmatrix}.$$

and

$$\mathscr{L}^{-1}\mathscr{J}\vec{\psi}_1 = \begin{pmatrix} 0 \\ 0 \\ -L_2^{-1}\varphi' \\ 0 \end{pmatrix}, \quad \mathscr{L}^{-1}\mathscr{J}\vec{\psi}_2 = \begin{pmatrix} L_2^{-1}\varphi \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathscr{L}^{-1}\mathscr{J}\vec{\psi}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -L_4^{-1}\varphi \end{pmatrix}.$$

We have

$$V = \begin{pmatrix} \langle L_2^{-1}\varphi', \varphi'\rangle & 0 & 0 \\ 0 & \langle L_1^{-1}\varphi, \varphi\rangle & 0 \\ 0 & 0 & \langle L_4^{-1}\varphi, \varphi\rangle \end{pmatrix}.$$

Since L_4 is strong positive, then $\langle L_4^{-1}\varphi, \varphi\rangle > 0$. The eigenfunction of L_2 corresponding to the negative eigenvalue is $\theta_0(\alpha x)$ and eigenfunction of L_2 corresponding to the zero eigenvalue is $\theta_1(\alpha x)$. Since $\varphi' \perp \{\theta_0, \theta_1\}$, then $\langle L_2^{-1}\varphi', \varphi'\rangle > 0$. Hence, the number of negative eigenvalues of matrix V depends on the sign of $\langle L_1^{-1}\varphi, \varphi\rangle$.

Now, we will compute $\langle L_1^{-1}\varphi, \varphi\rangle$. We will do it by constructing the Green function for the operator L_1. We already have $\varphi' \in Ker(L_1)$. The classical approach is to consider the function

$$\psi(x) = \varphi'(x)\int_0^x \frac{1}{\varphi'^2(s)}ds, \quad \begin{vmatrix} \varphi' & \psi \\ \varphi'' & \psi' \end{vmatrix} = 1$$

which is also solution of $L_1\psi = 0$. However, since φ' has zeros, the integral above is not well-defined. Instead, using the identities

$$\frac{1}{sn^2(y,\kappa)} = -\frac{1}{dn(y,\kappa)}\frac{\partial}{\partial_y}\frac{cn(x,\kappa)}{sn(y,\kappa)}$$

and integrating by parts, we get the equivalent formula

$$\psi(x)=\frac{1}{\alpha^2\varphi_0}\left[cn(\alpha x)-\alpha\kappa^2 sn(\alpha x,\kappa)dn(\alpha x,\kappa)\int_0^x\frac{1+cn^2(\alpha s,\kappa)}{dn^2(\alpha s,\kappa)}ds\right].$$

Thus, we may take the Green function in the form

$$L_1^{-1}f=\varphi'\int_0^x\psi(s)f(s)ds-\psi(s)\int_0^x\varphi'(s)f(s)s+C_f\psi(x),$$

where C_f is chosen such that $L_1^{-1}f$ is periodic with same period as $\varphi(x)$. After integrating by parts, we get

$$\langle L_1^{-1}\varphi,\varphi\rangle=-\langle\varphi^3,\psi\rangle+\frac{\varphi^2(T)+\varphi(0)^2}{2}\langle\varphi,\psi\rangle+C_\varphi\langle\varphi,\psi\rangle. \tag{13}$$

Similarly as in [9], integrating by parts yields

$$\langle\psi'',\varphi\rangle=2\psi'(T)\varphi(T)+\langle\psi,\varphi''\rangle.$$

Using that $L_1\varphi=-2\varphi^3$, we get

$$\langle\psi,\varphi^3\rangle=-\psi'(T)\varphi(T).$$

Now, integrating by parts and using the above relations, we get

$$\begin{aligned}\langle\varphi,\psi\rangle&=\tfrac{2}{\alpha^3(1-\kappa^2)}E(\kappa)\\ C_\varphi&=-\tfrac{\varphi''(T)}{2\psi'(T)}\langle\varphi,\psi\rangle+\tfrac{\varphi^2(T)-\varphi^2(0)}{2}.\end{aligned}\tag{14}$$

With this, we obtain

$$\langle L^{-1}\varphi,\varphi\rangle=-\frac{2}{\alpha}\frac{E^2(\kappa)-2(1-\kappa^2)E(\kappa)K(\kappa)+(1-\kappa^2)K^2(\kappa)}{(2\kappa^2-1)E(\kappa)+(1-\kappa^2)K(\kappa)}<0. \tag{15}$$

From here, we get that $n(V)=1$ and that the right side of (11) is odd number ($n(\mathscr{L})-n(V)=4-1=3$) and hence we have instability.

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