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NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF A POSITIVE DEFINITE SOLUTION OF A MATRIX EQUATION*

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ABSTRACT: *In this paper we study a matrix equation. We give a necessary and sufficient condition for the existence of a positive definite solution of the considered equation. We determine a set of matrices containing all positive definite solutions. Moreover, we propose a basic fixed point iteration for finding a positive definite solution. The theoretical results are illustrated by numerical examples.*

KEYWORDS: *Nonlinear matrix equation, positive definite solution*

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1 Introduction

We consider the nonlinear matrix equation

$$X - A^*XA - B^*X^{-1}B = I, \tag{1}$$

where A, B are $n \times n$ complex matrices, I is the identity matrix, and A^* denotes the conjugate transpose of A.

Eq. (1) has been introduced by Ali in [1] where an iterative method for computation a positive definite solution is proposed. In [2] by using the fixed point theorem for mixed monotone operator in a normal cone Gao has proved that the equation $X - A^*X^pA - B^*X^{-q}B = I$ with $0 < p, q < 1$ always has the unique positive definite solution. The equation $X - A^*XA + B^*X^{-1}B = I$ has been investigated in [3].

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Eq. (1) can be interpreted as a combination of the well-known equations $X - A^*XA = I$ [4, 5] and $X - B^*X^{-1}B = I$ [6, 7].

In addition, there are some contributions in the literature to the solvability and numerical solutions of the matrix equation $X + A^*X^{-1}A - B^*X^{-1}B = I$ [8, 9]. Konstantinov et al. [10] have investigated for the sensitivity of the equation $A_0 + \sum_{i=1}^{k} \sigma_i A_i^* X^{p_i} A_i = 0$, which is more general type of Eq. (1).

Motivated by [1, 2, 3], we study Eq. (1) for the existence of a positive definite solution and bounds of the solutions and iterative methods for obtaining of a solution. In addition, we consider some numerical examples to illustrate the theoretical results.

Throughout this paper, $\mathscr{C}^{n\times n}$ denotes the set of $n\times n$ complex matrices and $\mathscr{H}^n$ – the set of $n\times n$ Hermitian matrices. $A > 0$ $(A \geq 0)$ means that A is a Hermitian positive definite (semidefinite) matrix. If $A - B > 0$ (or $A - B \geq 0$) we write $A > B$ (or $A \geq B$). For $N \geq M > 0$ we use $[M, N]$ to denote the set of matrices $\{X : M \leq X \leq N\}$. We use $\rho(A)$ and $\|A\|$ to denote the spectral radius and the spectral norm $\left(\|A\| = \sqrt{\rho(A^*A)}\right)$ of a $n\times n$ matrix A, respectively.

2 A necessary and sufficient condition

Firstly, we will present some preliminaries.

Lemma 1. *[5] Let A, Q be square matrices.*

(a) If $\rho(A) < 1$, the Stain's equation $X - A^*XA = Q$ has a unique solution P_Q and $P_Q \geq 0$ $(P_Q > 0)$, when $Q \geq 0$ $(Q > 0)$.

(b) If there is some $P > 0$ such that $P - A^*PA$ is positive definite (semidefinite), then $\rho(A) < 1$ $(\rho(A) \leq 1)$.

In [1], it has been obtained the following necessary conditions for the existence of a positive definite solution and its lower bound.

Theorem 1. *[1, Theorem 2.1.] Let X be a positive definite solution of Eq. (1). Then* *(i)* $\rho(A) < 1$,

(ii) $\rho(X^{-1}B) < 1$,

(iii) $X \geq M$, *where* M *is the unique positive definite solution of the equation* $X - A^*XA = I$.

Now, we give a necessary and sufficient condition for the existence of a positive definite solution of Eq. (1) and an upper bound of all the solutions.

Theorem 2. *Eq. (1) has a positive definite solution* X, *if and only if* $\rho(A) < 1$. *Moreover, the all positive definite solutions are in* $[M,N]$, *where* M *and* N *are the unique solutions of the equations* $X - A^*XA = I$ *and* $X - A^*XA = I + B^*M^{-1}B$, *respectively.*

Proof: Let X be a positive definite solution of Eq. (1). Then by Theorem 1 it follows $\rho(A) < 1$.

Let $\rho(A) < 1$, then by Lemma 1 *(i)* the equation $X - A^*XA = Q$ has a unique positive definite solution for arbitrary $Q > 0$. Let M and N be the unique solutions of the equations $X - A^*XA = I$ and $X - A^*XA = I + B^*M^{-1}B$, respectively. Once again, by Lemma 1 *(i)* we have $M \leq N$.

Now, we consider a map F, defined by

$$F(X) = I + A^*XA + B^*X^{-1}B, \quad X > 0. \tag{2}$$

We will show that $F([M,N]) \subset [M,N]$. Let $X \in [M,N]$. Then

$$\begin{aligned} F(X) &= I + A^*XA + B^*X^{-1}B \\ &\geq I + A^*MA = M, \end{aligned}$$

$$\begin{aligned} F(X) &= I + A^*XA + B^*X^{-1}B \\ &\leq I + A^*XA + B^*X^{-1}B = N. \end{aligned}$$

Therefore, for all $X \in [M,N]$, $F(X) \in [M,N]$. Since $[M,N]$ is a convex, closed and bounded set and the map F is continuous on $[M,N]$, by Brouwer's fixed point theorem [11, p.17] it follows that there exists a solution $X \in [M,N]$ of Eq. (1). □

3 Basic fixed point iteration

In [1] Ali has investigated the iterative method

$$(3) \qquad \begin{cases} X_0 = I, \;\; Y_0 = \beta I, \;\; \beta > 1 \\ X_{k+1} = I + A^* X_k A + B^* Y_k^{-1} B, \quad k = 0, 1, \ldots \\ Y_{k+1} = I + A^* Y_k A + B^* X_k^{-1} B \end{cases}$$

for computing a positive definite solution of Eq. (1) based on the mixed monotone operator $G(X,Y) = I + A^*XA + B^*Y^{-1}B$. It was proven that the sequences $\{X_k\}$ and $\{Y_k\}$ defined in (3) with $\beta \geq \frac{1+\|B\|^2}{1-\|A\|^2}$ are convergent to a unique positive definite solution of Eq. (1) under condition $\|A\|^2 + \|B\|^2 < 1$. Moreover, $\{X_k\}$ and $\{Y_k\}$ have following properties:

$$(4) \qquad X_0 \leq X_1 \leq \cdots \leq X_k \leq Y_k \leq \cdots \leq Y_1 \leq Y_0.$$

We note that the iterative method (3) can be used with $X_0 = M$ and $Y_0 = N$, where the matrices M and N are from Theorem 2. From (4) we conclude that if $\lim_{k\to\infty} \|Y_k - X_k\| = 0$, then Eq. (1) has a unique positive definite solution.

We consider the basic fixed point iteration (BFPI):

$$(5) \qquad Z_{k+1} = I + A^* Z_k A + B^* Z_k^{-1} B, \quad k = 0, 1, \ldots, \;\; Z_0 \in [X_0, Y_0],$$

where X_0 and Y_0 are initial value in method (3).

We will prove that $X_k \leq Z_k \leq Y_k$ for all $k = 0, 1, \ldots$. We have $X_0 \leq Z_0 \leq Y_0$ by definition. Assume that $X_k \leq Z_k \leq Y_k$. Then

$$\begin{aligned} Z_{k+1} &= I + A^* Z_k A + B^* Z_k^{-1} B \\ &\leq I + A^* Y_k A + B^* X_k^{-1} B = Y_{k+1} \end{aligned}$$

and

$$\begin{aligned} Z_{k+1} &= I + A^* Z_k A + B^* Z_k^{-1} B \\ &\geq I + A^* X_k A + B^* Y_k^{-1} B = X_{k+1}. \end{aligned}$$

Therefore $X_k \leq Z_k \leq Y_k$ for all $k = 0, 1, \ldots$. Thus, we conclude that if $\lim_{k\to\infty} \|Y_k - X_k\| = 0$, then the BFPI (5) converges to a unique positive definite solution of Eq. (1).

4 Numerical experiments

In this section we carry out numerical experiments for computing the positive definite solutions of Eq. (1) by iterative methods (3) and (5) with $X_0 = M$, $Y_0 = N$ and $Z_0 = (M+N)/2$, respectively.

Let us $res(X) = \|X - A^*XA - B^*X^{-1}B - I\|_\infty$. As practical stopping criterions we use $\|Y_k - X_k\|_\infty \leq tol$ and $\|Z_k - Z_{k-1}\|_\infty \leq tol$ for methods (3) and (5), respectively, where k is the number of iterations.

We use the Matlab function *dlyap* for computing the unique positive definite solutions M and N of the equations $X - A^*XA = I$ and $X - A^*XA = I + B^*M^{-1}B$, respectively.

Example 1.[1] We consider Eq. (1) with matrix coefficients

$$A = \frac{1}{56}\begin{pmatrix} 1 & 5 & 3 & 2 \\ -1 & -6 & 3 & 4 \\ -4 & 3 & 7 & 5 \\ 1 & 8 & 2 & 1 \end{pmatrix}, \quad B = \frac{1}{70}\begin{pmatrix} 7 & 9 & 6 & 8 \\ 7 & 5 & 8 & 3 \\ 9 & 8 & 6 & 7 \\ 11 & 5 & 9 & 3 \end{pmatrix}.$$

In Table 1 we report the results of experiments for Example 1 with $tol = 10^{-10}$ by using iterative methods (3) and (5).

Example 2. We consider Eq. (1) with matrix coefficients

$$A = \frac{1}{200}\begin{pmatrix} 41 & 15 & 23 & 35 & 66 \\ 25 & 12 & 27 & 45 & 21 \\ 23 & 27 & 28 & 16 & 24 \\ 15 & 45 & 16 & 52 & 65 \\ 66 & 21 & 24 & 65 & 35 \end{pmatrix}, B = \frac{1}{30}\begin{pmatrix} 23 & 21 & 23 & 25 & 32 \\ 21 & 45 & 60 & 42 & 33 \\ 23 & 24 & 34 & 18 & 17 \\ 13 & 42 & 18 & 44 & 30 \\ 32 & 33 & 26 & 30 & 26 \end{pmatrix}.$$

Table 1: Numerical results for Example 1

(3)	BFPI (5)
$k = 11$	$k = 10$
$\|Y_{11} - X_{11}\|_\infty = 6.98e - 11$	$\|Z_{10} - Z_9\|_\infty = 5.89e - 11$
$res\left(\frac{X_{11}+Y_{11}}{2}\right) = 5.72e - 13$	$res(Z_{10}) = 5.53e - 12$

In Table 2 we report the results of experiments for Example 1 by using iterative methods (3) and (5) with $tol = 10^{-10}$ and $tol = 10^{-14}$, respectively.

Table 2: Numerical results for Example 2 for

(3)	BFPI (5)
$tol = 10^{-10}$	
$k = 169$	$k = 73$
$\|Y_{169} - X_{169}\|_\infty = 9.42e - 11$	$\|Z_{73} - Z_{72}\|_\infty = 9.73e - 11$
$res\left(\frac{X_{169}+Y_{169}}{2}\right) = 4.89e - 15$	$res(Z_{73}) = 7.06e - 11$
$tol = 10^{-14}$	
$k = 225$	$k = 102$
$\|Y_{225} - X_{225}\|_\infty = 9.77e - 15$	$\|Z_{102} - Z_{101}\|_\infty = 9.33e - 15$
$res\left(\frac{X_{225}+Y_{225}}{2}\right) = 1.78e - 15$	$res(Z_{102}) = 6.66e - 15$

The spectral radii of the matrices A in Example 1 and 2 are $\rho(A) = 0.1579$ and $\rho(A) = 0.8813$, respectively. For computing a positive definite solution of Eq. (1) with BFPI (5) we need less iterations than iterative method (3).

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