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BOURGAIN ALGEBRAS OF SOME IDEALS IN H^{∞} *

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ABSTRACT: *Let B be a Blaschke product having simple zeros and let I be a principal ideal in H^{∞} generated by B . In this article is shown that the Bourgain algebra of I relative to L^{∞} contains the Sarason algebra $H^{\infty}+C$. i.e. $\left(I,L^{\infty}\right)_b \supset H^{\infty}+C$*

KEYWORDS: *Bounded analytic functions; Bourgain algebras; Blaschke products; Ideals; Finitely generated ideals.*

1 Introduction and preliminaries

Let H^{∞} be the algebra of bounded analytic functions in the open unit disk D (with pointwise operations and the supremum norm) and $L^{\infty}\left(T\right)$ be the algebra of all essentially bounded, Lebesque measurable functions on the unit circle $\partial D = T$. Taking the boundary values of the functions on T we can consider H^{∞} as a closed subalgebra of $L^{\infty}\left(T\right)$. Its spectrum, or maximal ideal space, is the space $M(H^{\infty})$ of all nonzero multiplicative linear functionals on $H^{\infty}\left(D\right)$ endowed with the weak*- topology. Then $M(H^{\infty})$ is a compact Hausdorff space and the corona theorem says that D is dense in $M(H^{\infty})$ $\left[1\right]$. We denote the space of continuous functions on T by $C = C\left(T\right)$. Let z belongs to D and $\varphi_z\left(f\right) = f\left(z\right)$ for every $f \in H^{\infty}$. Then φ_z is a complex homomorphism "evaluation at the point z", i.e. $\varphi_z \in M\left(H^{\infty}\right) \subset \left(H^{\infty}\right)^{*}$.

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A sequence $\{z_n\}_n$ in D is called interpolating if for every bounded sequence $\{a_n\}_n$ of complex numbers there is a function $f \in H^\infty$ such that $f(z_n) = a_n$ for all n.

For a sequence $\{z_n\}_n$ in D with $\sum_{n=1}^{\infty}(1-|z_n|) < \infty$, the function:

$$B(z) = \prod_{n=1}^{\infty} \frac{-\overline{z}_n}{|z_n|} \frac{z - z_n}{1-\overline{z}_n z}, \; z \in D,$$

is called a Blaschke product with zeros $\{z_n\}_n$. If $\{z_n\}_n$ is an interpolating sequence, then $B(z)$ is also called interpolating. The study of interpolating sequences is useful in many areas of function theory and operator theory. Interpolating sequences can be applied for obtaining new scalar solutions of nonlinear differential equations using results in $[2]$ and $[3]$.

A closed subalgebra between H^∞ and L^∞ is called a Douglas algebra. By the Chang Marshall theorem $[1]$, every Douglas algebra A coincides with the closed subalgebra generated by H^∞ and complex conjugate of interpolating Blaschke products B with $\overline{B} \in A$. The Sarason algebra $H^\infty + C$ is a typical Douglas algebra and $H^\infty + C = \left[H^\infty, \overline{z}\right]$.

If X is any commutative algebra, we denote by

$$I = I(f_1, f_2, ..., f_N) = \left\{\sum_{i=1}^{N} h_i f_i : h_i \in X\right\}$$

the ideal generated by the f_i. If N can be chosen to be one, then I is call a principal ideal. We will say that I is radical ideal if $f \in I$ whenever some power f^n of $f \in X$ belong to I.

In $[4]$ it was shown that in H^∞ a radical ideal $I \neq (0)$ is finitely generated if and only if I is a principal ideal generated by a Blashke product having simple zeros.

Let Y be a Banach algebra and X be a linear subspace of Y. The Bourgain algebra X_b or $(X,Y)_b$ of X relative to Y is defined to the set of all $f \in Y$ such that:

if $f_n \to 0$ weakly in X, then $\operatorname{dist}\left(f.f_n, X\right) \to 0$.

The distance, $\operatorname{dist}\left(f.f_n, X\right)$ between $f.f_n$ and X is the quotient norm of the coset $f.f_n + X$ in the space Y/X. J. Cima and R. Timony $[5]$ proved that: X_b is a closed subalgebra of Y and contains the constant functions; if X is an algebra then $X \subset X_b$. In $[6]$ J. Cima, Sv. Janson and K. Yale showed that that the Bourgain algebra of H^{∞} relative $L^{\infty}(T)$ is $H^{\infty}+C$. P. Gorkin, K. Izuchi and R. Mortini $[7]$ present another proof. They also prove many properties of the Bourgain algebras in the case $Y = L^{\infty}(T)$ and X - closed subalgebra between H^{∞} and $L^{\infty}(T)$.

In this paper we prove that Bourgain algebra of I relative to L^{∞}, contains the Sarason algebra i.e. $\left(I, L^{\infty}\right)_b \supset H^{\infty}+C$, where I is principal ideal in H^{∞} generated by Blashke product, having simple zeros.

2 The main result

Theorem 2.1. Let I be a principal ideal in H^{∞} generated by Blaschke product B having simple zeros. The Bourgain algebra of I relative to L^{∞} contains the Sarason algebra $H^{\infty}+C$, i.e. $\left(I, L^{\infty}\right)_b \supset H^{\infty}+C$.

Proof: Since $I = BH^{\infty}$ is an algebra, the space $\left(I, L^{\infty}\right)_b$ is a closed subalgebra of L^{∞} and $I \subset \left(I, L^{\infty}\right)_b$. If $f \in H^{\infty}$ then $f.Bg \in H^{\infty}$ for every $g \in H^{\infty}$ and we obtain that $H^{\infty} \subset \left(I, L^{\infty}\right)_b$.

(i) First we will look at the case when $B(0) \neq 0$.

Let $f_n \to 0$ weakly in $I = BH^{\infty}$. Since BH^{∞} is contains in H^{∞} therefore $f_n \to 0$ weakly in H^{∞}, i.e. $\varphi(f_n) \to 0$ for all φ in $\left(H^{\infty}\right)^*$. For $\varphi = \varphi_0$ we have $\varphi_0(f_n) = f_n(0) \to 0$. If $f_n = Bg_n$ where $g_n \in H^{\infty}$ then $f_n(0) = B(0)g_n(0)$. But $B(0) \neq 0$ and we obtain $g_n(0) \to 0$.

Put $t_n(z) = g_n(z) - g(0)$. Since

$$\overline{z}t_n = \left(g_n(z) - g_n(z)\right).\overline{z} = \left(g_n(z) - g_n(z)\right)/z$$

for $z \in T$ and $\left(g_n(z) - g_n(z)\right)/z \in H^\infty$ then $\overline{z}t_n \in H^\infty$. Hence

$$dist\left(\overline{z}f_n, I\right) = dist\left(\overline{z}Bg_n, BH^\infty\right) = dist\left(\overline{z}g_n, H^\infty\right) = dist\left(\overline{z}t_n + \overline{z}g_n(0), H^\infty\right) =$$

$$dist\left(\overline{z}g_n(0), H^\infty\right) = \inf\left\{\|h\|_\infty : h \in \left[\overline{z}g_n(0)\right]\right\} \le \|\overline{z}g_n(0)\| = |g_n(0)| \to 0$$

and $\overline{z}$ belongs to $\left(I, L^\infty\right)_b$.

(ii) Now let $B(0) = 0$. Since B is a Blaschke product having simple zeros we may consider that $B = zb$ and b is a Blaschke product such that $b(0) \ne 0$. If $f_n \to 0$ weakly in $I = BH^\infty$, then $f_n \to 0$ weakly in bH^∞, because $I = BH^\infty = zbH^\infty \subset bH^\infty$. If $f_n = Bg_n$ where $g_n \in H^\infty$ we have:

$$dist\left(\overline{z}f_n, I\right) = dist\left(\overline{z}Bg_n, I\right) = dist\left(\overline{z}Bg_n, zbH^\infty\right) = dist\left(\overline{z}^2 Bg_n, bH^\infty\right) \underset{n\to\infty}{\to} 0$$

as using the case (i) $\overline{z}^2$ belongs in $\left(bH^\infty, L^\infty\right)_b$. Therefore $\overline{z}$ belongs to $\left(I, L^\infty\right)_b$.

Since z and $\overline{z}$ belong to the closed subalgebra $\left(I, L^\infty\right)_b$ of L^∞, then by the Weierstrase theorem we have $C \subset \left(I, L^\infty\right)_b$. The theorem is proved.

Corollary 2.2. Let $I \ne (0)$ be a finitely generated radical ideal in H^∞. Then $\left(I, L^\infty\right)_b \supset H^\infty + C$

Proof: By [3] I is a principal ideal generated by e Blashke product having simple zeros and we can apply **Theorem 2.1.**

In the case where I is a principal ideal in H^∞ generated by a finite Blaschke product it can be proven that $\left(I, L^\infty\right)_b \supset H^\infty + C$.

We need two lemmas.

Lemma 2.3. $[1]$. If $\{z_n\} \subset D$ is interpolating sequence, then there exist functions $\{f_n\} \subset H^\infty$ and positive number M such that $f_n(z_n) = 1$ for all n, $f_n(z_k)=0$ for $n\neq k$ and $\sum_{n=1}^{\infty}|f_n(z)|\leq M$ for all $z\in D$.

Lemma 2.4. $[6]$. Suppose that $\{f_n\}$ is a sequence in H^∞ such that $\sum_{n=1}^{\infty}|f_n(z)|\leq M$ for all $z\in D$. Then $f_n\to 0$ weakly in H^∞.

Theorem 2.5. Let I be a principal ideal in H^∞ generated by a finite Blaschke product B. Then $(I,L^\infty)_b = H^\infty + C = (H^\infty, L^\infty)_b$.

Proof: Since B is a finite Blaschke product, then zB and $\overline{zB}$ belongs to $H^\infty + C = (H^\infty, L^\infty)_b$ $[1]$. If $f_n\to 0$ weakly in $BH^\infty\subset H^\infty$ then $f_n\to 0$ weakly in H^∞ and we obtain:

$$dist(\overline{z}f_n, I) = dist(\overline{z}f_n, BH^\infty) = dist(\overline{zB}.f_n, H^\infty)\to 0\text{, i.e. } \overline{z}\in(I,L^\infty)_b.$$

Since $H^\infty\subset(I,L^\infty)_b$ this means $H^\infty + C = (H^\infty, L^\infty)_b\subset(I,L^\infty)_b$.

Hence $(I,L^\infty)_b$ is a Douglas algebra which contains $[H^\infty,\overline{z}]$. By the Chang Marshall theorem every Douglas algebra A such that $[H^\infty,\overline{z}]\subsetneq A$ is generated by H^∞ and complex conjugate of infinite interpolating Blaschke products. To show equality is sufficiently to prove that $(I,L^\infty)_b$ does not contain the complex conjugate of any infinite interpolating Blaschke product.

Let q be an interpolating Blaschke product and let $\{z_n\}\subset D$ denote the zero sequence of q. According to lemma 1.1 there exist functions $\{f_n\}\subset H^\infty$ and positive number M such that $f_n(z_n)=1$ for all n, $f_n(z_k)=0$ for $n\neq k$ and $\sum_{n=1}^{\infty}|f_n(z)|\leq M$ for all $z\in D$. Then for functions $g_n(z)=B(z)f_n(z)$ we obtain:

$$g_n\in BH^\infty = I;\ g(z_n) = B(z_n)f_n(z_n) = B(z_n)\text{ for all } n;$$

$$g_n\left(z_k\right)=0 \text{ for } n\neq k \text{ and } \sum_{n=1}^{\infty}\left|g_n\left(z\right)\right|\leq M \text{ for all } z\in D.$$

By lemma 1.2 (with BH^∞ instead of H^∞) we have $g_n\to 0$ weakly in $BH^\infty = I$ but:

$$\operatorname{dist}\left(\bar{q}.g_n,\ I\right)=\operatorname{dist}\left(g_n, q.I\right)=\inf\left\{\sup_{z\in D}\left|g_n(z)-q(z).y(z)\right| : y\in I\right\}\geq$$

$$\geq \inf_{z\in D}\left\{\left|g_n\left(z_n\right)-q\left(z_n\right)y\left(z_n\right)\right| : y\in I\right\}=\left|B\left(z_n\right)\right|$$

and $\left|B\left(z_n\right)\right|$ do not turn to zero, because B is a finite Blaschke product.

Thus $\bar{q}\notin\left(I,L^\infty\right)_b$, and the theorem is proved.

Since z^pH^∞ is an algebra, the space $\left(z^pH^\infty\right)_b$ is a closed subalgebra of L^∞ and $z^pH^\infty\subset\left(z^pH^\infty\right)_b$. If $f\in H^\infty$ then $f.z^pg\in z^pH^\infty$ for every $g\in H^\infty$ and we obtain that $H^\infty\subset\left(z^pH^\infty\right)_b$

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