REMARKS ON HYPERCOMPLEX ANTI-CIRCULANT ARITHMETICS AND GEOMETRY

STANCHO G. DIMIEV

I am very grateful to the Organizing Committee of the Scientific Conference “MATTEX 2014” in Shumen University, and especially to the Dean of Mathematical Departament Prof. Dimcho Stankov an to Prof. Wejdi Hassanov, inviting me to take part in this Conference. As I am born in Popovo, near Shumen, this is a pleasure for me to visit my native land.

Now I come to the general motivation of my talk. It is related with different question of actual interest in mathematics and mathematical physics. The principal one is about the mathematical description of the notion of space-time in terms of complex geometry, following Russian mathematician Jury Manin (coherent sheaves, spinors etc.) [6], instead of the language of differential geometry (gauge theory etc.) [7].

For this purpose I develop an extension of the classical complex geometry to a hypercomplex geometry. In this talk I concern some preliminaries only.

I hope these Remarks will be accepted as a (non systematic) exposition of some ideas written ad hoc for a short time.

&1. MULTICOMPLEX IMAGINARY ARITHMETIC: NOTATION AND BASIC DEFINITIONS.

Traditional notation for the natural numbers N, for the fields of real numbers R and complex numbers C, and the quaternion noncomutative body H are used in the next $\mathbb{N} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

The classical multicomplex imaginary theory is based on the linear notation $x_0 + x_1 i_1 + x_2 i_2 + ... + x_n i_n$ where $x_k \in \mathbb{R}$ and $i_k{}^2 = -1;\ k = 1, 2, ..., n$. (to see Kantor amd Solodownokov [1]) But we shall consider a kind of polynomial imaginary arithmetics using analogous notation $Z = z_0 + jz_1 + j^2 z_2 + ... + j^m z_m$, where

$z_k \in \mathbb{C}, k = 0,1,2,...,m,\ m+1 = 2^n,\ j^{m+1} = -1$. In fact Z is a matrix called anti-circulant. [2]

It is important to remark that the basic imaginary unit is defined as the matrix below

$$j = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

The other imaginary units are defined as degrees of the basic one j, i.e. $j^2, j^3, ..., j^m, j^{m+1} = -1$ are special matrices of integer coefficients.

Definition

The matrix Z is an anti-circulant matrix [2] called hypercomplex number of order 2^n. [3] The set of all such hypercomplex numbers is denoted by $\mathbb{C}(1, j, j^2, j^3, ..., j^m)$, i.e.

$$\mathbb{C}(1, j, j^2, j^3, ..., j^m) = \{Z : Z = z_0 + jz_1 + j^2 z_2 + ... + j^m z_m,\ z_k \in \mathbb{C}\}$$

In the paper [3] we use an equivalent form for the above definition of anti-circulant matrix.

The particular case of $m=1$, i.e. $\mathbb{C}(1, j), j^2 = -1$, was considered by Hamilton himself without publishing, and by Italian mathematician Corrado Segre, who introduce the term "bicomplex number", "bicomplex algebra" (to see the paper of Segre in Mathematische Annalen [5]).

It is to remark that two Bulgarian mathematicians Prof. Emanuil Ivanov and Prof. L. Tschakalov studied bicomplex algebra around 1920-23, publishing in Bulgarian [6].

The set $\mathbb{C}(1, j, j^2, j^3, ..., j^m)$ can be equipped with algebraic operations : addition and multiplication induced by imaginary basis $(j^k . j^l = j^{k+l} / \sim)$.

So this set can be considered as a $\mathbb{C}$-algebra (more precisely as a coordinate $\mathbb{C}$-algebra). The general case of functional $\mathbb{C}$-algebras is obtained considering complex functions $f(z)$, where $z = (z_0, z_1, z_2, ..., z_m)$.

&2. SINGULARITIES IN HYPERCOMPLEX ANTI-CIRCULANT ARITHMETICS.

The arithmetics of these hypercomplex numbers is of specific character. Let us compare the vector space $\mathbb{C}^{m+1}$ with the $\mathbb{C}$-algebra $\mathbb{C}(1,j,j^2,\dots j^m)$. The advantage of $\mathbb{C}(1,j,j^2,\dots j^m)$ is in circumctanse that it admits a naturally defined multiplications, derived directly from the imaginary basis $(1,j,j^2,\dots j^m)$. But this is on the prize of the acceptance of singular elements. Recall that in the field $\mathbb{C}$ there is only one singular element – the zero complex number.

In fact hypercomplex number Z, $Z = z_0 + jz_1 + j^2 z_2 + ... + j^m z_m$, admits anti-circulant matrix representation, denoted M(Z). More precisely

$$M(z) = \begin{pmatrix} z_0 & z_1 & z_2 & \cdots & z_m \\ -z_m & z_0 & z_1 & \cdots & z_{m-1} \\ -z_{m-1} & -z_m & z_0 & \cdots & z_{m-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -z_1 & -z_2 & -z_3 & \cdots & z_0 \end{pmatrix}$$

Finaly we obtain the chain of transformation
$z \mapsto \mathbb{Z} \mapsto M(\mathbb{Z})$,
starting with a vector $z = (z_0, z_1, z_2, ..., z_m)$ in $\mathbb{C}^{m+1}$.

Definition

SyngLocus ($\mathbb{C}$(1,j,j^2,…j^m))=$\{Z : \det M(Z) = 0\}$

In the particular case (m = 1) of bicomplex algebra $\mathbb{C}$(1,j) we have

SingLocus($\mathbb{C}$(1,j))= $\left\{ z_0 + jz_1 : \det\begin{pmatrix} z_0 & z_1 \\ -z_1 & z_0 \end{pmatrix}, j^2 = -1 \right\}$

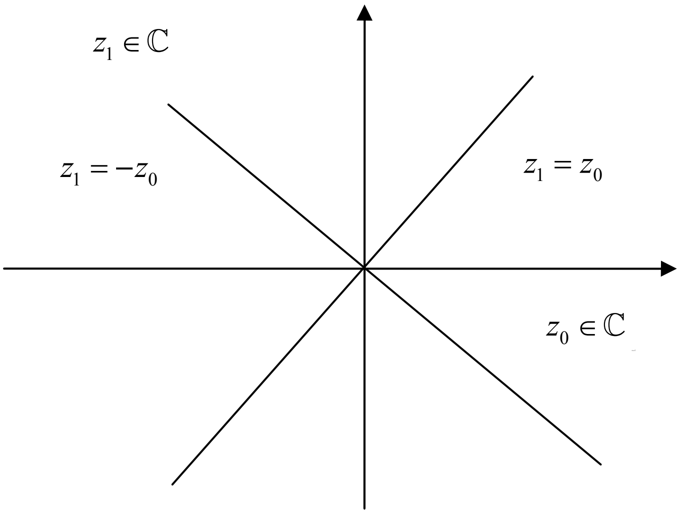
This means that we have the equatinon

$$z_0^2 + z_1^2 = 0$$

as a characteristic of singular bicomplex coordinating.

In view that $z_0^2 + z_1^2 = (z_0 + jz_1)(z_0 - jz_1)$ we find the solution of the above written equation, namely the pairs (z_0, z_0) and $(z_0, -z_0)$, or $(z_1 = z_0)$ and $(z_1 = -z_0)$.

Geometricaly we obtain the pair of complex bisectrices in the bicomplex plane $\mathbb{C}^2$



This is an abstract picture of cross-shaped form the set of bicomplex singularities.

One can consider the case of 4-complex number

$$Z = \begin{pmatrix} z_0 & z_1 & z_2 & z_3 \\ -z_3 & z_0 & z_1 & z_2 \\ -z_2 & -z_3 & z_0 & z_1 \\ -z_1 & -z_2 & -z_3 & z_0 \end{pmatrix} = (z_0 + j^2 z_2) + j(z_1 + j^2 z_3)$$

,

$$\det\begin{pmatrix} (z_0 + j^2 z_2) & (z_1 + j^2 z_3) \\ -(z_1 + j^2 z_3) & (z_0 + j^2 z_2) \end{pmatrix} = (z_0 + j^2 z_2)^2 + (z_1 + j^2 z_3)^2,$$

As j^4=-1 we receive finally

$$\det Z = z_0^2 + z_1^2 - (z_2^2 + z_3^2) + 2j(z_0 z_2 + z_1 z_3),$$

so, the equation of singular 4-complex number reduces to the system

$$\left| \begin{array}{l} z_0^2 + z_1^2 - (z_2^2 + z_3^2) = 0 \\ \quad z_0 z_2 + z_1 z_3 = 0 \end{array} \right.$$

In the general case of the singularities let
$Z \in \mathbb{C}(1, j, j^2, j^3, ..., j^m), \quad m+1 = 2^n, \quad n \in \mathbb{N}$
$Z = z_0 + jz_1 + j^2 z_2 + ... + j^m z_m$
One set
$Z_0 = z_0 + j^2 z_2 + ... + j^{m-1} z_{m-1}$
$Z_1 = z_1 + j^2 z_3 + ... + j^{m-1} z_m$
Clearly $Z = Z_0 + jZ_1, \quad Z^* = Z_0 - jZ_1$, then $ZZ^* = Z_0^2 - j^2 Z_1^2$

Apllying this formula for bicomplex number one receive $ZZ^* = z_0^2 + z_1^2, \quad j^2 = -1$

So in the general case Z is called singular if $ZZ^* = 0$, for bicomplex number we receive the equation for singularities $z_0^2 + z_1^2 = 0$,

Remark. In the field $\mathbb{C}$ if $z = x + iy$, $z^{-1} = \dfrac{\overline{z}}{x^2 + y^2}$, $x^2 + y^2 \neq 0$.

&3. A POSSIBLE APPLICATION

The proof of the fundamental theorem of algebra for polynomial with complex coefficients depends of the fact that the algebra $\mathbb{C}$ is a field, but this circumstance does not holds for bicomplex algebra $\mathbb{C}(1, j), \ j^2 = -1$. It is not a field, because it is non-division algebra.

An attractive question is the following one: Can one prove a kind of equivalent form of the fundamental theorem for bicomplex algebra.

We shall consider polynomial with bicomplex coefficients

$$P(x)=\sum_{k=0}^{p}(a_k+jb_k)X^k,\ \ j^2=-1\ ,\ a_k\in\mathbb{C}, b_k\in\mathbb{C}, k=0,1,2....,p$$

Clearly we have

$P(x)=P_0(x)+jP_1(x)$

Where $P_0(x)$ and $P_1(x)$ are polynomial with complex coefficients

$$P_0(x)=\sum_{k=0}^{p}a_kX^k\ \ ,\ \ P_1(x)=\sum_{k=0}^{p}b_kX^k$$

Here X is a formal variable. We shall consider complex variable $X=z_0$, and bicomplex variable $X=z_0+jz_1,\ \ j^2=-1$

The notion of a resultant $R(P_0,P_1)$, is taken from the classical elimination theory for the complex case $X=z_0$. In this case the condition $R(P_0,P_1)=0$, Implies that $P_0(x)$ and $P_1(x)$ has a common complex root.

It is interesting, for the bicomplex variable $X=z_0+jz_1,\ \ j^2=-1$, to answer the above state assertion, including if the common root is a regular or singular element of bicomplex algebra.

The general case of order 2^n,

$\mathbb{C}(1,j,...,j^m)\left[X_1,...,X_p\right]$

seems to be very difficult technically. It seems too that it shall be useful to introduce a notion of projective bicomplex and multycomplex spaces.

REFERENCES

1. I. L. Kantor and A. S. Solodovnikov, Hypercomplex Numbers - an Elementary Introduction to Algebras, Springer, Berlin, 1989.
2. Jun-qing Wang and Li-zheng Zhang, Partition Anticirculant Matrix and Skew Field and its Qualities, Intern. Journal of Algebra, vol. 3 No 7, 325-329, (2009)
3. L.N.Apostolova, St. Dimiev, M.S.Marinov and P.Stoev, Problems in hypercomplex analysis over some matrix hypercomplex systems. (AMEE`11, Sozopol, 2011)
4. S.Corado, Le representazikoni reali delle forme complesse ed gli enti iperalgebici, Math. Anallen (1892) 101-109
5. Ljubomir Tchakalov, On tetranions of E.Ivanov, Ann. De l`Universite de Sofia, Fac.de Phys. Mat., 20, (1924), 89-120 (in Bulgarian, with German abstracts)
6. Ю.И. Манин, Калибровочные поля и комплексная геометрия М. 1984
7. T.Equichi, P.B.Qilkey, I.Y.Hanson, Gravitation, gauge theory and differential geometry, Phys. Rep. 1980, vol.66,No6, 213-393.
8. S.Lang, Algebra Addison-Wesley, 1965