MATTEX 2018,
CONFERENCE PROCEEDING, v. 1, pp. 213- 220

INTERLACING MATHEMATICS AND ART IN THE CLASSROOM: TEACHING SYMMETRY AND ANTISYMMETRY USING TRUCHET TILES*

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ABSTRACT: *The construction of art works based on a module — modular art — is present throughout all human history. Naturally, modularity is a fertile field for the occurrence of symmetries. In this paper we consider the use of a particular module consisting of a square divided by one of its diagonals into two triangles of different colors, known as a Truchet tile. From this module we considered rosettes and friezes of different dimensions and studied their properties regarding possible symmetries and antisymmetries. We have also counted the different configurations that can be obtained for different dimensions of rosettes and friezes. Symmetry and antissymmetry can be useful when creating art works and interlacing mathematics with the arts in the classroom can be a successful way to promote the interest for both subjects. We present some works carried out by primary and secondary school students and teachers.*

KEYWORDS: *Modular art, Symmetry, Antisymmetry, Truchet tiles, Mathematics and art, Mathematics education.*

1 Introduction

Modular structures are structures constructed from a set of basic elements (modules). The principle of modularity manifests itself in many ways, in nature, in science, in art, etc. As Slavik Jablan says [2], "modularity is a manifestation of the universal principle of economy in nature: the possibility of diversity and variability of structures, resulting from some sets (finite and very restricted) of basic elements, through their recombination". Of course, when recombination is based on isometries, we often find several symmetries in this type of structures.

In 1704, a Dominican priest named Sébastien Truchet published a work "Memoir sur les Combinasions", where he explored the construction of patterns made from a simple module composed of a square divided by one of its diagonals into two triangles of different colors, ◪, now known as a Truchet tile. Later, in 1722, a colleague of Truchet, Father Dominique Doüat, published the book "Méthode pour faire une infinité de dessins différents, avec des carreaux mi-partis de deux couleurs par une ligne diagonale", pursuing the work of Truchet and considering many other patterns constructed with the same motif. The work of these two priests became known through the much more recent publication of Smith and Bouchet [6] that triggered the interest for patterns made out of Truchet tiles.

Truchet tiles allow not only to tile the plane but also to create very appealing rosettes and friezes. These tilings, friezes and rosettes can be found in a variety of real-life applications, including patchwork, tapestries and facades of buildings. In Portugal, in the beginning of the 19th century, several facades of buildings were covered with blue and white Truchet tiles. In the city of Oporto there are more than 40 buildings with these tiles. Figure 1 shows three examples of facades with three distinct patterns that are found in Oporto together with a fourth, less symmetrical pattern found on a facade panel of the *Ginásio Clube Português*, in Lisbon.

*This work was supported in part by the Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia), through CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013.

Figure 1: Photos of three buildings in Oporto, from googlemaps, and detail (right) from the tile panel on the facade of the *Ginásio Clube Português*, in Lisbon.

In Greece, in a small village on the island of Chios called Pyrgi, there are many ornamental friezes of Truchet tiles (or a non-square rectangular version of it) on the facades of the houses. Figure 2 shows two examples.

Figure 2: Photos from houses in Pyrgi, Chios, Greece.

Truchet tile are also used by artists as can be seen in a ceramic panel by the Portuguese ceramist Sofia Beça shown in Figure 3.

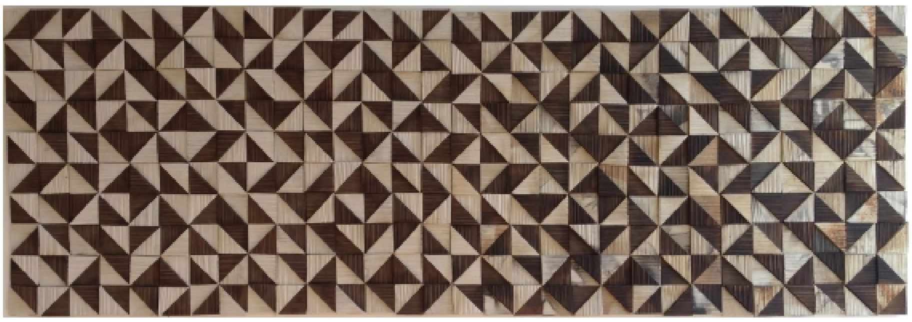

Figure 3: Ceramic panel “Regresso às origens” (“Back to the origins”) by Sofia Beça, 2016.

2 Symmetry and antisymmetry

In geometry, a symmetry of a figure is an isometry that leaves it invariant. The set of symmetries of a figure F together with the operation composition forms a group which is known as the symmetry group of F. A symmetry group can either be discrete or continuous. Most figures we are interested in have discrete groups. In the plane there are only three categories of discrete symmetry groups: rosette groups (they have a finite number of symmetries which can only be rotations or reflections); frieze groups (they have translation symmetry in only one direction; one can identify a motive which is replicated at a constant distance along a straight line); and wallpaper groups (they have translation symmetry in two directions and spread over the plane). In this paper we shall focus on rosettes and friezes. Rosettes can be only of two types: either their symmetry group is cyclic, C_n, in which case they have exactly n rotations as symmetries (including the identity as a trivial rotation) or their group is dihedral, D_n, in which case they have exactly n rotations and n reflections as symmetries. Figure 4 contains a set of examples of rosettes from both types.

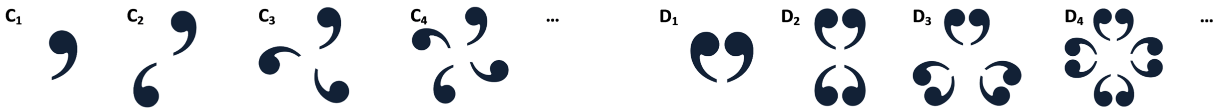

Figure 4: Rosette symmetry groups.

Friezes may have seven different symmetry groups, resulting from all the possible combinations of the four symmetry types (rotational, reflection, translational and glide reflection). Figure 5 contains a set of examples of friezes from all types (we use the crystallography notation). For more detailed information on symmetry groups see for instance [4].

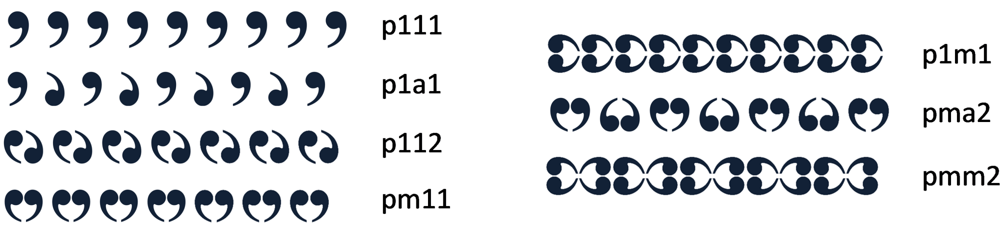

Figure 5: Frieze symmetry groups.

Humans have a natural tendency to identify symmetries around them - it is part of our way of perceiving the world and processing the information we are constantly receiving through the eyes. It is therefore not surprising that throughout human history human creations contain numerous elements of symmetry. But too much symmetry becomes monotonous and breaking symmetry is as important as symmetry itself. One was to break it is through antisymmetry; the concept is simple and appealing and can help students assimilate and consolidate the concept of symmetry.

Antisymmetry (also known as two-color symmetry) is closely connected to the concept of symmetry and may occur whenever each point of a figure or object has associated a dichotomous characteristic such as one of two colors, one of two electric charges, etc.. An antisymmetry is simply a symmetry coupled to an exchange of colors (or exchange of the value of the dichotomous variable) that leaves the figure or object invariant. Since there are four possible types of symmetry on the plane, there are also four possible types of antisymmetry on the plane. The yin-yang symbol, ☯, is an example of a figure with rotational antisymmetry and no rotational symmetry.

The set of all antisymmetries and symmetries of a figure also forms a group. Antisymmetry groups can be derived from the symmetry groups by coupling a permutation group with only two elements, the color-change transformation (see [5] for more details on antisymmetry groups). There are two practical issues that are useful for understanding the antisymmetry groups of a figure or even for generating such groups.

1. The designations for the symmetry groups can be obtained by analyzing the symmetry group of the uncolored figure (only with contours), G_u, and the symmetry group of the colored figure, G_c. The name of the anti-symmetry group is G_u/G_c. Figure 6 illustrates this process.

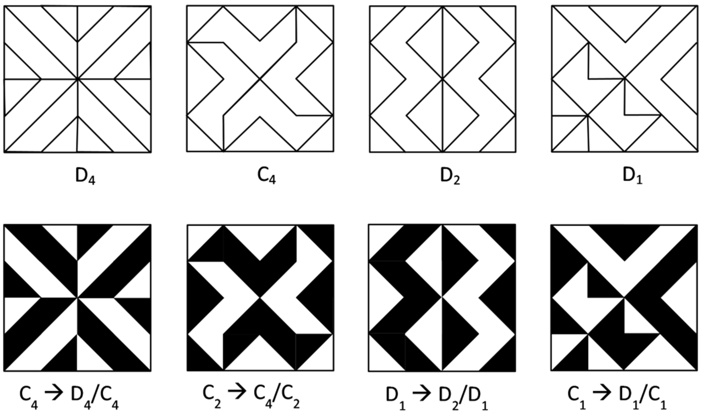

Figure 6: Examples of antisymmetry classification.

2. To generate an antisymmetry group we can start from a symmetry group and substitute one (or more) of its generators by the corresponding antisymmetric one. For example, group D_4 (symmetry group of a square) can be generated by a rotation of 90° and a reflection. If we replace one of these generators with the respective antisymmetric we obtain an antisymmetry group.

3 Truchet rosettes and friezes

Lord and Ranganatan [3] challenged the academic community to find the number of different patterns that can be constructed with Truchet tiles, given a unit cell dimension. In order to respond to this challenge, we began to count rosettes and friezes of Truchet tiles. Table 1 contains the counts of some square rosettes of Truchet tiles.

nxn	1x1	2x2	3x3	4x4	5x5	6x6
D_4		2		16		512
C_4		1		120		130816
D_2		2		240		261632
C_2		2		16200		17179672832
D_1	1	12	256	65280	16777216	68719214592
C_1	0	24	32640	536830080	1,407E14	5,903E20
Tot.	1	43	32896	536911936	1,407E14	5,903E20

Table 1: Counts of square Truchet rosettes by symmetry group.

As can be see the number of rosettes grows very rapidly with the square dimensions. One can also see that (in general) the more symmetrical the rosette the fewer configurations there are. As an example we provide all the 2×2 truchet rosettes grouped by symmetry and antisymmetry groups in Figure 7.

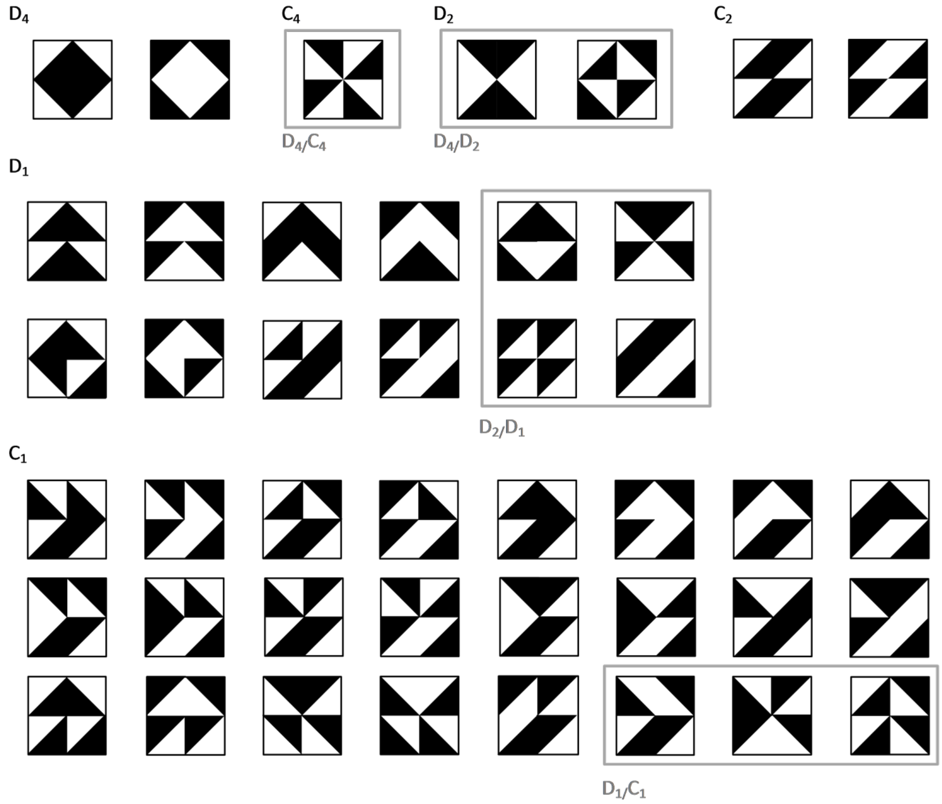

Figure 7: 2×2 Truchet rosettes and their symmetry groups (in black) amd antisymmetry groups (in grey).

With respect to friezes we provide in Figure 8 an example with all the six possible friezes with cell dimensions 2×1. There is only one 1×1 Truchet frieze, three 1×2 friezes and thirty eight 2×2 friezes.

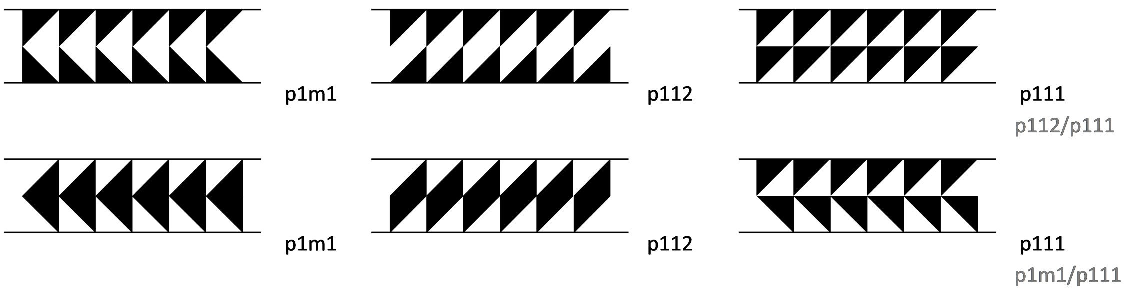

Figure 8: 1×2 Truchet friezes and their symmetry groups (in black) amd antisymmetry groups (in grey).

4 Mathematics and art in the classroom

Symmetry and antissymmetry can be useful when creating art works. Interlacing mathematics with the arts in the classroom can be a successful way to promote the interest for both subjects. Elliot Eisner (1933-2014), a pioneer in arts education, believed that an artistic approach to education could improve its quality and lead to a new vision for teaching and learning [1]. The study of symmetry and isometry, present throughout the school mathematics curriculum from elementary to secondary levels makes a perfect setting for a deeper contribution of art to mathematics education. The use of Truchet tiles for creating patterns is a simple way for mathematics teachers to explore mathematical concepts and artistic creativity with their students.

In this section we present some applications carried out in an educational context, with students and teachers of primary and secondary education. During the last six years several professional development courses took place at the University of Aveiro involving mathematics and arts teachers. In these courses several topics from geometry have been addressed, including symmetry (in rosettes, friezes and wallpaper patterns) and antissymetry. In all courses teachers had a chance to deepen their knowledge on the subjects involved and were asked to create their own art works applying those subjects. In turn, teachers were asked to perform some tasks with their students and challenge them to create their own art works. Parallel to these courses several activities have taken place directly with students, such as summer schools at University for elementary and secondary school students. The applications given next were all conducted within these setups.

In 2015, twenty students from the fifth and sixth grades attended the mathematics summer school at the University of Aveiro. One of the activities consisted of exploring Truchet friezes. Students were challenged to find out all the possible Truchet friezes with cell size 2×2. They couldn't find all the 38 friezes but they found most of them. Then they had to describe their symmetries and group the friezes according to the symmetry type. Finally each student decorated the front and back cover of an A4 size notebook with a different pair of Truchet friezes. Figure 9 contains some of the resulting friezes.

Figure 9: Friezes on notebooks by fifth and sixth grade students (2015).

In 2017, during a workshop on mathematics and art for fifth to twelfth grades mathematics teachers, the participants were asked to freely create a 4×4 Truchet rosette. Figure 10 contains the resulting rosettes. All together there were 17 rosettes but two were identical (upper left rosette).

Afterwards teachers classified the symmetry group and realized that most of their rosettes had some kind of symmetry or regularity which was not expectable if they had just chosen their configurations at random. At this point we explored some statistics and compared the observed frequencies with the expected ones. This activity served, among other things, to show that humans tend to look for symmetry in what surrounds them and that reflection symmetry arises more naturally than rotational one.

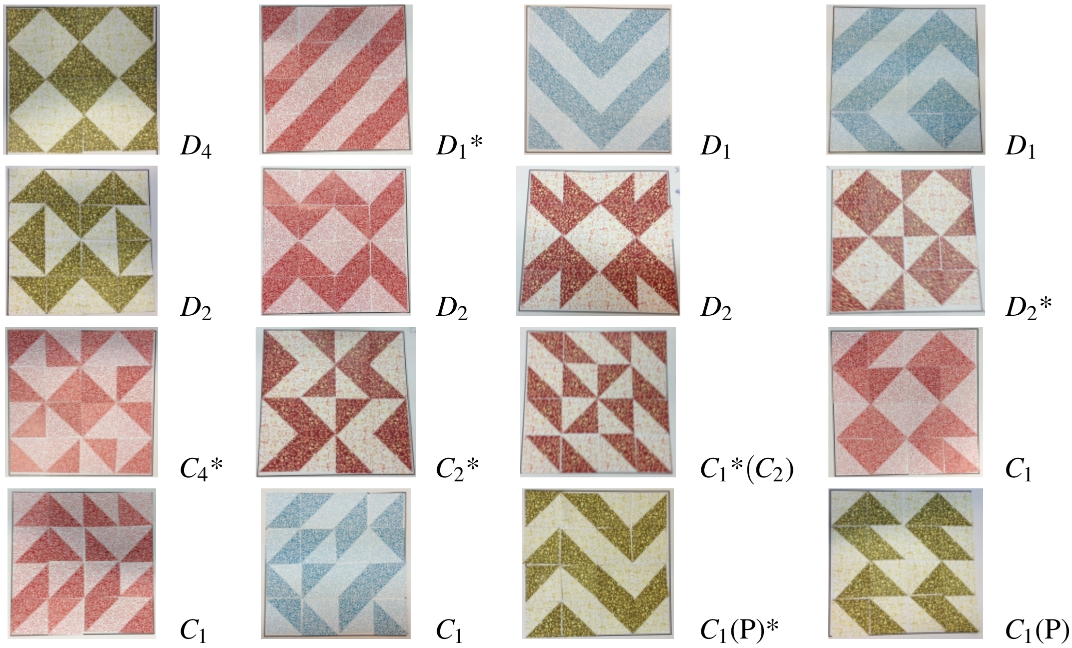

Figure 10: 4×4 Truchet rosettes made by teachers (2017). (P) - suggests a wallpaper; * - with antisymmetry; (C_2) - suggests C_2.

In 2017 two professional courses took place where teachers explored the concept of antisymmetry. Figures 11 to 13 show some of the results produced both by the teachers and their students.

Figure 11: Patchwork and ceramic pieces exploring antisymmetry; created by Ana Paula Moreira (left; C_4/C_2) and Teresa Carvalho (center and right; $p112/p112$ and $pmm2/p112$) (2017).

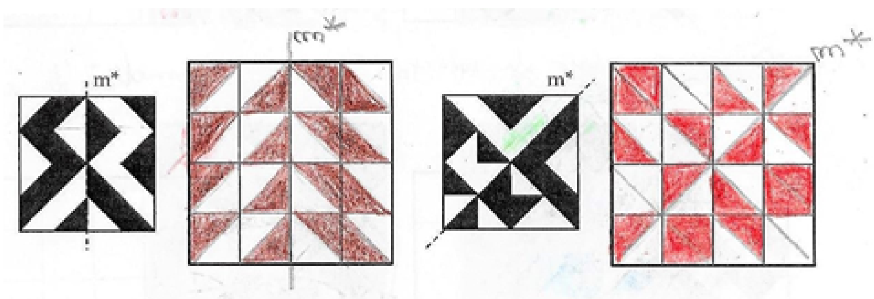

Figure 12: Antisymmetry explored by an elementary school student (2017).

Figure 13: Truchet rosettes created by elementary school students (2017).

More recently, in 2018, a professional development course with elementary school teachers addressed the topic of antisymmetry and some third grade students created interesting figures such as those in Figure 14 (Can you spot the flaw in the middle rosette?).

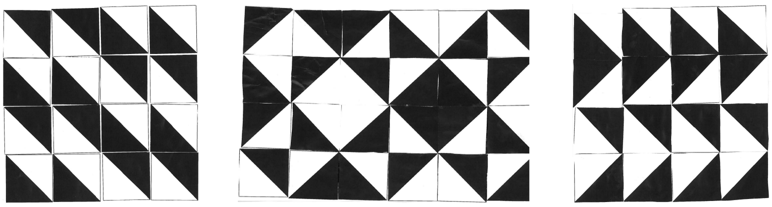

Figure 14: Truchet rosettes with antisymmetry created by third grade students (2018).

The professional development courses described above have been very rewarding, as participants reinforce their knowledge in mathematics and at the same time acquire new knowledge in arts and, above all, have the opportunity to explore their creativity and artistic skills, creating an extremely pleasant and productive work environment. In these courses we achieved the ultimate goal of all education, which is to successfully learn or teach with pleasure and satisfaction.

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