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ON P-GROUPS HAVING A NORMAL ELEMENTARY ABELIAN SUBGROUP OF INDEX P*

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ABSTRACT: *In this article we prove a classification theorem for p-groups G having a normal elementary abelian group H of index p, under the assumption that the p-th lower central subgroup $G_{(p)}$ is trivial.*

KEYWORDS: *nilpotent groups, p-groups.*

Let us first introduce some notations. The cyclic group of order n we denote by C_n. Let G be a group. The subgroups $G_{(0)} = G$ and $G_{(i)} = [G, G_{(i-1)}]$ for $i \geq 1$ are called the lower central series of G. Let $G = \langle H, \alpha \rangle$, where H is a normal abelian subgroup of G and $\alpha^p \in H$. Denote $H^p = \{h^p : h \in H\}$ and $H(p) = \{h \in H : h^p = 1, h \notin H^p\}$. For any $\beta \in H$ define $N_G(\beta) = \langle \alpha^{-x} \beta \alpha^x : 0 \leq x \leq p-1 \rangle$, the normalizer of $\langle \beta \rangle$ in G.

We are going to prove now some properties concerning the generators and relations in the p-groups G having a normal abelian group H of index p under the assumption that $G_{(p)} = \{1\}$. Bender started the study of these groups in [1], and the first author investigated the related Noether's problem in [2, 3].

Lemma 1. *Let p be a prime, let $n \geq 2$ and let G be a group of order p^n with a normal abelian subgroup H of order p^{n-1}. Then H is subject to the following conditions:*

1. *For any $\beta_1 \in H, \beta_1 \notin Z(G)$, there exist $\beta_2, \dots, \beta_k \in H$ for some $k \geq 2$ such that $[\beta_j, \alpha] = \beta_{j+1}$, where $1 \leq j \leq k-1$ and $\beta_k \neq 1$*

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is central. We call k the length of the commutator chain, starting with β_1.

2. *The normalizer $N_G(\beta_1)$ is generated by $\beta_1, \beta_2, \ldots, \beta_k$.*

3. *$N_G(\beta_1) = N_G(\beta_1')$ for $\beta_1' \in N_G(\beta_1)$ if and only if $\beta_1' = \prod_{i=1}^{k} \beta_i^{x_i}$ for some integers x_i, where $\gcd(x_1, p) = 1$.*

4. *Let $\beta = \prod_{i=2}^{k} \beta_i^{x_i}$ for some integers x_i, such that $\gcd(x_{i_0}, p) = 1$, where $i_0 = \min\{i : x_i \neq 0\}$. Then β appears in a commutator chain starting with a generator β_1', i.e., there exist $\beta_1', \cdots, \beta_{i_0}' = \beta$ such that $[\beta_j', \alpha] = \beta_{j+1}'$, for $1 \leq j \leq i_0 - 1 \leq k-1$, and $N_G(\beta_1) = N_G(\beta_1')$. In particular, if $\beta_2^p = 1$, then every element $\beta \in \langle \beta_2, \ldots, \beta_k \rangle$ appears in a commutator chain starting with a generator β_1' such that $N_G(\beta_1) = N_G(\beta_1')$.*

Proof. (1) Since G is nilpotent, there exist $\beta_2, \ldots, \beta_k \in H$ for some $k \geq 2$ such that $[\beta_j, \alpha] = \beta_{j+1}$, where $1 \leq j \leq k-1$ and $\beta_k \neq 1$ is central.

(2) We have that $\beta_2 = [\beta_1, \alpha] = \alpha^{-0}\beta_1^{-1}\alpha^0 \cdot \alpha^{-1}\beta_1\alpha \in N_G(\beta_1)$, and for any $x : 3 \leq x \leq k$ we make the inductive assumption that $\beta_{x-1} = [\beta_1, \alpha^{x-2}]h$ for $h \in \langle \alpha^{-y}\beta_1\alpha^y : 0 \leq y \leq x-3 \rangle \leq N_G(\beta_1)$. Using the identity $[\beta_1, \alpha^{x-1}] = [\beta_1, \alpha^{x-2}][\beta_1, \alpha][[\beta_1, \alpha^{x-2}], \alpha]$ we see that
$[[\beta_1, \alpha^{x-2}], \alpha] = [\beta_{x-1}h^{-1}, \alpha] = \beta_x[h^{-1}, \alpha] =$
$= [\beta_1, \alpha^{x-1}][\beta_1, \alpha^{x-2}]^{-1}[\beta_1, \alpha]^{-1}$. Hence
$\beta_x = [\beta_1, \alpha^{x-1}]h_1 \in N_G(\beta_1)$, where $h_1 = [\beta_1, \alpha^{x-2}]^{-1}[\beta_1, \alpha]^{-1}[h, \alpha] \in \langle \alpha^{-y}\beta_1\alpha^y : 0 \leq y \leq x-2 \rangle \leq N_G(\beta_1)$. Since $N_G(\beta_1)$ is the minimal normal subgroup of G containing β_1, we obtain that $N_G(\beta_1) = \langle \beta_1, \ldots, \beta_k \rangle$.

(3) Let $\beta_1' = \prod_{i=1}^{k} \beta_i^{x_i}$ for some integers x_i, where $\gcd(x_1, p) = 1$. From (2) it follows that $N_G(\beta_1) \geq N_G(\beta_1')$. On the other hand, $[\beta_1', \alpha^{k-1}] = \beta_k^{x_1} \in N_G(\beta_1'), [\beta_1', \alpha^{k-2}] = \beta_{k-1}^{x_1}\beta_k^{x_2} \in N_G(\beta_1'), \ldots,$
$[\beta_1', \alpha] = \prod_{i=2}^{k} \beta_i^{x_{i-1}} \in N_G(\beta_1')$ and $\beta_1' = \prod_{i=1}^{k} \beta_i^{x_i} \in N_G(\beta_1')$. Therefore, $\beta_1, \ldots, \beta_k \in N_G(\beta_1')$, i.e., $N_G(\beta_1) \leq N_G(\beta_1')$.

Conversely, if $N_G(\beta_1) = N_G(\beta_1')$, we have that $\beta_1' = \prod_{i=1}^{k} \beta_i^{x_i}$ and $\beta_1 = \prod_{i=1}^{k} \beta_i'^{y_i}$ for $\beta_{i+1}' = [\beta_i', \alpha]$ and some integers x_i, y_i. Therefore, $\beta_1 = \beta_1^{x_1 y_1} \prod_{i=2}^{k} \beta_i^{x_i y_1} \prod_{i=2}^{k} \beta_i'^{y_i}$, which is possible only if $x_1 y_1 \equiv 1 \pmod{\operatorname{ord}(\beta_1)}$, and hence $\gcd(x_1, p) = 1$.

(4) We can write $\beta = \prod_{i=i_0}^{k} \beta_i^{x_i} = \prod_{i=i_0}^{k} [\beta_{i-1}^{x_i}, \alpha] = [\prod_{i=i_0}^{k} \beta_{i-1}^{x_i}, \alpha]$. Put $\beta_{i_0-1}' = \prod_{i=i_0}^{k} \beta_{i-1}^{x_i}$, so $[\beta_{i_0-1}', \alpha] = \beta$. If $i_0 = 2$ then $N_G(\beta_1) = N_G(\beta_1')$ (according to (3)) and $[\beta_1', \alpha] = \beta$. If $i_0 > 2$ then $\beta_{i_0-1}' = [\beta_{i_0-2}', \alpha]$ for $\beta_{i_0-2}' = \prod_{i=i_0}^{k} \beta_{i-2}^{x_i}$. It is clear now that proceeding in this way we will obtain at the end a generator β_1' such that $N_G(\beta_1) = N_G(\beta_1')$ and $[\beta_j', \alpha] = \beta_{j+1}'$ for $1 \le j \le i_0 - 1$ and $\beta_{i_0}' = \beta$.

Finally, if $\beta_2^p = 1$, then from $[\beta_j^p, \alpha] = \beta_{j+1}^p$ it follows that $\beta_3^p = \cdots = \beta_k^p = 1$. Clearly, every non-trivial element $\beta \in \langle \beta_2, \ldots, \beta_k \rangle$ can be written as $\beta = \prod_{i=2}^{k} \beta_i^{x_i}$ for some integers x_i, such that $\gcd(x_i, p) = 1$. □

Next, we are going to consider the special case when H is an elementary abelian group.

Theorem 2. *Let p be prime, let $n \ge 2$ and let G be a group of order p^n with an abelian normal subgroup $H \simeq (C_p)^{n-1}$, the elementary abelian group of order p^{n-1}. Then H is subject to the following conditions:*

1. *There exist elements $\gamma_1, \ldots, \gamma_s \in H$ (for some $s \le n-1$) such that $H \simeq N_G(\gamma_1) \times \cdots \times N_G(\gamma_s)$;*

2. *For any $i : 1 \le i \le s$ there exists a natural number $k_i \le p$ such that $N_G(\gamma_i) \simeq (C_p)^{k_i}$, and generators $\beta_{i1}, \ldots, \beta_{ik_i} \in N_G(\gamma_i)$, such that $\beta_{i1} = \gamma_i, [\beta_{ij}, \alpha] = \beta_{ij+1}$ for $1 \le j \le k_i - 1$ and β_{ik_i} is central in G.*

Proof. First, let us decompose H as a direct product of cyclic groups: $H = \langle \gamma_1 \rangle \times \cdots \times \langle \gamma_{n-1} \rangle$. Then there exists a generator, say γ_1, such that $\gamma_1 \notin [H, \alpha]$. Indeed, if we suppose that $\gamma_i \in [H, \alpha]$ for all i, from the commutation rule $[a,\alpha][b,\alpha]=[ab,\alpha]$ we get that any element $\gamma=\prod_i \gamma_i^{a_i}\in H$ is in $[H,\alpha]$. But then we will obtain infinite commutator series $[\alpha_{j+1},\alpha]=\alpha_j, \alpha_1=\gamma_1, j=1,2,\ldots$, which is a contradiction, since G being a p-group is nilpotent.

Note that, since any group of order p^2 is abelian, $\langle\gamma_1\rangle$ is normal in G if and only if γ_1 is central in G. Clearly, if γ_1 is central in G, then $N_G(\gamma_1)=\langle\gamma_1\rangle\simeq C_p$. If all elements of H are central, then G is abelian and $s=n-1$.

Now, for any $i:1\le i\le n-1$ such that γ_i is not central in G, put $\beta_{i1}=\gamma_i$. Since G is nilpotent, there exist $\beta_{i2},\ldots,\beta_{ik_i}\in H$ for some $k_i\ge 2$ such that $[\beta_{ij},\alpha]=\beta_{ij+1}$, where $1\le j\le k_i-1$ and $\beta_{ik_i}\ne 1$ is central. Since α^p is in H, we have that

$$\beta_{i1}=\alpha^{-p}\beta_{i1}\alpha^p=\beta_{i1}\beta_{i2}^{\binom{p}{1}}\beta_{i3}^{\binom{p}{2}}\cdots\beta_{ip}^{\binom{p}{p-1}}\beta_{ip+1}=\beta_{i1}\beta_{ip+1},$$

therefore $\beta_{ip+1}=1$. Then $G_{(p)}=\{1\}, k_i\le p$ and $\beta_{ik_i+1}=\cdots=\beta_{ip+1}=1$. According to Lemma 1 we get $N_G(\gamma_i)=\langle\beta_{i1},\ldots,\beta_{ik_i}\rangle$.

Next, without loss of generality we may assume that γ_1 has the maximal length of the commutator chain, i.e., $k_1\ge k_j$ for $2\le j\le n-1$. Put $\beta_1=\beta_{11}=\gamma_1,\beta_2=\beta_{12},\ldots,\beta_{k_1}=\beta_{1k_1}$. Observe that the elements $\beta_1,\ldots,\beta_{k_1}$ are independent generators of $N_G(\gamma_1)$. Indeed, if we suppose they are dependent, then $\beta_k=\prod_{i<k}\beta_i^{x_i}$ for some $0\le x_i<p, 1\le k\le k_1$. Forming the commutator chain of β_k, we see that there exists a commutator that can be decomposed as a product of $\beta_k^{x_j}$ (for some $x_j\ne 0$) and powers of β_i for $i\ne k$. Thus we will get an endless commutator chain, which is impossible, G being nilpotent.

If $k_1<n-1$ then there exists another generator, say $\gamma_2\notin N_G(\gamma_1)$. We can also assume that γ_2 is not central, otherwise $H=N_G(\gamma_1)\times\langle\gamma_2\rangle\times\cdots\times\langle\gamma_s\rangle$ and we are done. Put $\beta_{k_1+1}=\beta_{21}=\gamma_2,\beta_{k_1+2}=\beta_{22},\ldots,\beta_{k_1+k_2}=\beta_{2k_2}$. Again, the elements $\beta_{k_1+1},\ldots,\beta_{k_1+k_2}$ are independent generators of $N_G(\gamma_2)$. However, it is possible that $\beta_{k_1+1},\ldots,\beta_{k_1+k_2}$ are dependent modulo $N_G(\gamma_1)$, i.e., $\prod_{i=1}^{k_1}\beta_i^{x_i}\prod_{i=k_1+1}^{k_2}\beta_i^{x_i}=1$ for $x_i:0\le x_i\le p-1$ such that $x_{i_0}\ne 0, x_{j_0}\ne 0$ for some $i_0,j_0:1\le i_0\le k_1, k_1+1\le j_0\le k_1+k_2$. If

we suppose that $x_{j_1} \neq 0$ for some $j_1 \neq j_0, k_1+1 \leq j_1 \leq k_1+k_2$ we will obtain that $\beta_{j_1} = \beta_{j_0}^{x} \cdots$ for $x \neq 0$ which leads to an endless commutator chain. Thus the only possibility is that there exists $\ell_2 \leq k_2$ such that $\beta_{k_1+1}, \ldots, \beta_{k_1+\ell_2-1} \notin N_G(\gamma_1)$, but $\beta_{k_1+\ell_2} \in N_G(\gamma_1)$.

According to Lemma 1 (4), $\beta_{2\ell_2} = \beta_{k_1+\ell_2}$ appears in a commutator chain starting with a generator γ_1' such that $N_G(\gamma_1) = N_G(\gamma_1')$, i.e., there exist $\beta_{11}', \ldots, \beta_{1\ell_1}' \in N_G(\gamma_1')$, such that $\beta_{11}' = \gamma_1', [\beta_{1j}', \alpha] = \beta_{1j+1}'$ for $1 \leq j \leq \ell_1 - 1$ and $\beta_{1\ell_1}' = \beta_{2\ell_2}$ for some $\ell_1 \leq k_1$. Notice that $k_1 - \ell_1 = k_2 - \ell_2$, because after $\beta_{2\ell_2}$ the two commutator chains coincide. Since we assumed that $k_1 \geq k_2$, we get $\ell_1 \geq \ell_2$. Define $\gamma_2' = \beta_{1\ell_1-\ell_2+1}'^{-1}\gamma_2$ and $\beta_{21}' = \gamma_2', [\beta_{2j}', \alpha] = \beta_{2j+1}'$ for $1 \leq j \leq \ell_2 - 1$. Therefore, $\beta_{2\ell_2}' = 1$ and $\beta_1, \ldots, \beta_{k_1}, \beta_{21}', \ldots, \beta_{2\ell_2-1}'$ are independent generators of the whole subgroup $N_G(\gamma_1)N_G(\gamma_2')$.

If $k_1 + \ell_2 - 1 < n - 1$ then there exists another generator, say $\gamma_3 \notin N_G(\gamma_1)N_G(\gamma_2')$ and we may proceed in a similar manner. Namely, suppose that for some $t \geq 2$ there exist generators $\gamma_2', \ldots, \gamma_t'$ such that $\beta_{i1}' = \gamma_i', [\beta_{ij}', \alpha] = \beta_{ij+1}'$ for $1 \leq j \leq \ell_i - 1, 2 \leq i \leq t$ and $\prod_{i=2}^{t}\prod_{j=1}^{\ell_j-1} \beta_{ij}'^{x_{ij}} \notin N_G(\gamma_1) \setminus \{1\}$ for any $0 \leq x_{ij} \leq p-1$. We proved this assertion for $t = 2$, and we will show that it holds for $t+1$.

Assume that $\gamma_{t+1} \notin N_G(\gamma_1)N_G(\gamma_2') \cdots N_G(\gamma_t')$ and $\prod_{i=2}^{t}\prod_{j=1}^{\ell_i-1} \beta_{ij}'^{x_{ij}} \cdot \prod_{j=1}^{k_{t+1}} \beta_{t+1j}^{x_{t+1j}} \in N_G(\gamma_1) \setminus \{1\}$. If we suppose that $x_{t+1j_0} \neq 0$ and $x_{t+1j_1} \neq 0$ for some $1 \leq j_0 < j_1 \leq k_{t+1}$ we will obtain a contradiction with the nilpotency of G. Thus the only possibility is that there exists $\ell_{t+1} \leq k_{t+1}$ such that $\beta_{t+1j} \notin N_G(\gamma_1)N_G(\gamma_2') \cdots N_G(\gamma_t')$ for $1 \leq j \leq \ell_{t+1} - 1$, where $\beta_{t+11} = \gamma_{t+1}, \beta_{t+1j+1} = [\beta_{t+1j}, \alpha]$, but $\beta_{t+1\ell_{t+1}} = \beta \prod_{i=2}^{t}\prod_{j=1}^{\ell_i-1} \beta_{ij}'^{y_{ij}}$, where $\beta \in N_G(\gamma_1) \setminus \{1\}$.

According to Lemma 1 (4), β appears in a commutator chain starting with a generator γ_1'' such that $N_G(\gamma_1) = N_G(\gamma_1'')$, i.e., there exist $\beta_{11}'', \ldots, \beta_{1\ell_1'}'' \in N_G(\gamma_1'')$, such that $\beta_{11}'' = \gamma_1'', [\beta_{1j}'', \alpha] = \beta_{1j+1}''$ for $1 \leq j \leq \ell_1' - 1$ and $\beta_{1\ell_1'}'' = \beta$ for some $\ell_1' \leq k_1$. Notice that $k_1 - \ell_1' \leq k_{t+1} - \ell_{t+1}$. (Here we might have an inequality, when $\beta_{t+1\ell_{t+1}+k_1-\ell_1'} \notin Z(G)$.)

Since we assumed that $k_1 \geq k_{t+1}$, we get $\ell'_1 \geq \ell_{t+1}$. Define $\gamma'_{t+1} = \beta''^{-1}_{1\ell'_1-\ell_{t+1}+1}\gamma_{t+1}$ and $\beta'_{t+11} = \gamma'_{t+1}, [\beta'_{t+1j}, \alpha] = \beta'_{t+1j+1}$ for $1 \leq j \leq \ell_{t+1} - 1$. Therefore, $\beta'_{t+1\ell_{t+1}} = \prod_{i=2}^{t}\prod_{j=1}^{\ell_i}\beta'^{y_{ij}}_{ij} \in N_G(\gamma'_2)\cdots N_G(\gamma'_t)$ and our assertion is proved.

We can continue this process until we finish the generators of H. Thus we will obtain finally that $H = N_G(\gamma_1)N_G(\gamma'_2)\cdots N_G(\gamma'_s)$ for some generators $\gamma_1, \gamma'_2, \ldots, \gamma'_s$ of direct cyclic factors such that $N_G(\gamma_1) \cap (N_G(\gamma'_2)\cdots N_G(\gamma'_s)) = \{1\}$. Therefore, $H = N_G(\gamma_1) \times (N_G(\gamma'_2)\cdots N_G(\gamma'_s))$, and we can apply induction on $N_G(\gamma'_2)\cdots N_G(\gamma'_s)$ (which, of course, is normal in G) to finish the proof. □

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