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RECOVERING SPACE CURVES BY MÖBIUS INVARIANTS*

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ABSTRACT: *One subgroup of the Möbius group in the Euclidean 3-space $\mathbb{R}^3$ is induced by the group of rotations on the unit 3-sphere in the Euclidean 4-space $\mathbb{R}^4$ via a stereographic projection. We find two functions as Möbius invariants and they determine a Frenet space curve up to a transformation of considered group. We use these invariants to construct space curves and for that purpose we apply an algorithm in the computer system Mathematica.*

KEYWORDS: *Möbius group, Frenet space curves, Stereographic projection*

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1 Introduction

The three dimensional unit sphere $\mathbb{S}^3$, centered at the origin O and embedded in the four-dimensional Euclidean space $\mathbb{R}^4$, is mapped via the stereographic projection π with a center at a point $P \in \mathbb{S}^3$ onto an equatorial hyperplane $\xi \subset \mathbb{R}^4$ through the origin O with a normal vector $\overrightarrow{OP}$. We consider the hyperplane ξ as a three-dimensional Euclidean space $\mathbb{R}^3$. The group of rigid motion on the sphere $\mathbb{S}^3$ coincides with the group of rotations $SO(4)$ in $\mathbb{R}^4$ about the origin O. This group induces in $\mathbb{R}^3$ via the stereographic projection π a subgroup M_0 of the Möbius group Möb(3) in $\mathbb{R}^3$. For that group it is naturally to apply the algebra of quaternions $\mathbb{H}$, where $\mathbb{H}_0 \subset \mathbb{H}$ is the set of pure quaternions (with zero real parts). If $w \in \mathbb{H}$ is a quaternion then the norm of w is denoted by

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$\|w\|$, the quaternion $\overline{w} \in \mathbb{H}$ is the conjugate quaternion of w and $\|w\|^2 = w.\overline{w} = \overline{w}.w$. The transformations from the group M_0 can be represented by the quaternion formalism

$$F(q) = (A.q + B).(B.q + A)^{-1},$$

where $A, B \in \mathbb{H}$, $q \in \mathbb{H}_0$, $\|A\|^2 + \|B\|^2 = 1$ and $A.\overline{B} + B.\overline{A} = 0$, identifying $\mathbb{H}_0$ with $\mathbb{R}^3$ (Theorem 3 in [2]).

It is well-known that the Euclidean curvature $\varkappa$ and the Euclidean torsion τ determine the space curves up to orientation preserving motions in $\mathbb{R}^3$. This fundamental theorem in the classical curve theory is extended to the orientation preserving similarity group in [1]. Recovering a space curve by a pair of real functions up to an orientation preserving motion or a similarity transformation is based on the mentioned theorems. The Euclidean geometry of curves on $\mathbb{S}^3 \subset \mathbb{R}^4$ is carried naturally on $\mathbb{R}^3$ by the stereographic projection π and it can be interpreted with respect to the subgroup $M_0 \subset$Möb(3) in $\mathbb{R}^3$. The differential geometry of curves with respect to the Möbius group is developed by Udo Hertrich-Jeromin in [7]. In the presented paper it is used a different approach. Relations between the Euclidean differential-geometric invariants of curves on the sphere $\mathbb{S}^3$ and the corresponding curves in $\mathbb{R}^3$ via the stereographic projection π are found.These relations determine invariants of curves with respect to the group M_0. They have an essential roll of recovering the curves in $\mathbb{R}^3$ up to transformations from the considered group. The case of plane curves, recovering by one Möbuis invariant, is explored in [3] and [4].

2 Preliminaries

Let $K = O\vec{e}_1\vec{e}_2\vec{e}_3\vec{e}_4$ be a right-handed Cartesian coordinate system in $\mathbb{R}^4$. For any two vectors $\boldsymbol{x} = (x^1, x^2, x^3, x^4)$ and $\boldsymbol{y} = (y^1, y^2, y^3, y^4)$ in $\mathbb{R}^4$ the canonical inner product $< \boldsymbol{x}, \boldsymbol{y} >$ is defined by $< \boldsymbol{x}, \boldsymbol{y} >= \sum_{i=1}^{4} x^i y^i$. We denote the norm of the vector $\boldsymbol{x}$ by $\|\boldsymbol{x}\| = \sqrt{< \boldsymbol{x}, \boldsymbol{x} >}$. The inner product and the norm of vectors in $\mathbb{R}^3$ are determined in a similar way.

Let $\mathbb{S}^3 = \{(x^1,x^2,x^3,x^4) \in \mathbb{R}^4 | (x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2 = 1\}$ be the unit sphere in $\mathbb{R}^4$, centered at the origin O, and let $\boldsymbol{l} = (x^1,x^2,x^3,x^4), x^1 \neq 1$ be the position vector of an arbitrary point on $\mathbb{S}^3$, different from the pole $P(1,0,0,0)$. Denoting by (u^1,u^2,u^3) the coordinates of any position vector $\boldsymbol{u}$ of an arbitrary point in the equatorial hyperplane $\xi \equiv \mathbb{R}^3$ with respect to the Cartesian coordinate system $K' = O\vec{e}_1'\vec{e}_2'\vec{e}_3'$ in $\mathbb{R}^3$, where $\vec{e}_1' = \vec{e}_2$, $\vec{e}_2' = \vec{e}_3$, $\vec{e}_3' = \vec{e}_4$, the stereographic map $\pi : \mathbb{S}^3 \backslash \{P\} \to \mathbb{R}^3$ is given by the equalities $u^i = \dfrac{x^{i+1}}{1-x^1}$, $i = 1,2,3$. For the reverse map π^{-1} we have the equalities

$$\begin{aligned} x^1 &= \frac{(u^1)^2+(u^2)^2+(u^3)^2-1}{(u^1)^2+(u^2)^2+(u^3)^2+1}, \\ x^i &= \frac{2u^{i-1}}{(u^1)^2+(u^2)^2+(u^3)^2+1}, \; i = 2,3,4. \end{aligned}$$

Making the infinity ∞ correspond to the pole P, the stereographic projection π and the reverse map π^{-1} are bijective conformal maps.
We consider the parametric equations of the sphere $\mathbb{S}^3$ in the form

$$\boldsymbol{l}(u^1,u^2,u^3) = \left\{ \frac{\|\boldsymbol{u}\|^2-1}{\|\boldsymbol{u}\|^2+1}, \frac{2u^1}{\|\boldsymbol{u}\|^2+1}, \frac{2u^2}{\|\boldsymbol{u}\|^2+1}, \frac{2u^3}{\|\boldsymbol{u}\|^2+1} \right\}.$$

The standard orthogonal tangential frame field $\left\{ \dfrac{\partial \boldsymbol{l}}{\partial u^1}, \dfrac{\partial \boldsymbol{l}}{\partial u^2}, \dfrac{\partial \boldsymbol{l}}{\partial u^3} \right\}$ at any point of the sphere $\mathbb{S}^3$, except the point P, is given by

$$\begin{aligned} \boldsymbol{l}_1 = \frac{\partial \boldsymbol{l}}{\partial u^1} &= \mu . \left\{ u^1, \frac{1-(u^1)^2+(u^2)^2+(u^3)^2}{2}, -u^1u^2, -u^1u^3 \right\} \\ \boldsymbol{l}_2 = \frac{\partial \boldsymbol{l}}{\partial u^2} &= \mu . \left\{ u^2, -u^1u^2, \frac{1+(u^1)^2-(u^2)^2+(u^3)^2}{2}, -u^2u^3 \right\} \\ \boldsymbol{l}_3 = \frac{\partial \boldsymbol{l}}{\partial u^3} &= \mu . \left\{ u^3, -u^1u^3, -u^2u^3, \frac{1+(u^1)^2+(u^2)^2-(u^3)^2}{2} \right\}, \end{aligned}$$

where $\mu = \dfrac{4}{(1+\|\boldsymbol{u}\|^2)^2}$ and $\boldsymbol{l}_1^2 = \boldsymbol{l}_2^2 = \boldsymbol{l}_3^2 = \mu$.

A regular space curve in $\mathbb{R}^3$ with a nowhere vanishing Euclidean curvature is called a Frenet space curve. For a given Frenet space curve $c : \boldsymbol{u}(s) = (u^1(s), u^2(s), u^3(s))$ in $\mathbb{R}^3$, parameterized by an arc-length parameter s, we denote by "$'$" the differentiation with respect to s. Applying the Frenet equations for the derivatives (see [5]), it is easy to prove the equalities

$$(1)\qquad \begin{gathered} <\boldsymbol{u}', \boldsymbol{u}'''> = -<\boldsymbol{u}'', \boldsymbol{u}''> = -\varkappa^2, \\ <\boldsymbol{u}'', \boldsymbol{u}'''> = \left(\frac{\boldsymbol{u}''^2}{2}\right)' = \varkappa\varkappa', <\boldsymbol{u}' \times \boldsymbol{u}''', \boldsymbol{u}''> = -\varkappa^2\boldsymbol{\tau}, \end{gathered}$$

where $\varkappa$ and $\boldsymbol{\tau}$ are the Euclidean curvature and the Euclidean torsion of c, respectively.

Lemma 2.1. *Let $\boldsymbol{v} = \boldsymbol{v}(s)$ be a vector function in $\mathbb{R}^3$ with coordinate functions $v^k(s) = \sum_{i,j=1}^3 \Gamma_{ij}^k u^{i'} u^{j'}$, $k = 1,2,3$, where Γ_{ij}^k are the Christoffel symbols for the sphere $\mathbb{S}^3$ and $u^i = u^i(s)$, $i = 1,2,3$ are the coordinate functions of the unit speed curve $c : \boldsymbol{u} = \boldsymbol{u}(s)$. Then*

$$\boldsymbol{v} = \sqrt{\mu}.\boldsymbol{u} - \mu.\left(\frac{1}{\mu}\right)'.\boldsymbol{u}'$$

and

$$<\boldsymbol{v}, \boldsymbol{u}'''> = \varkappa^2\mu\left(\frac{1}{\mu}\right)' - \sqrt{\mu}\left(\frac{\sqrt{\mu} + \frac{\mu^2}{4}\left(\frac{1}{\mu}\right)'^2 - \frac{\mu}{2}\left(\frac{1}{\mu}\right)''}{\sqrt{\mu}}\right)',$$

$$\boldsymbol{v}^2 = \mu\boldsymbol{u}^2, <\boldsymbol{v}, \boldsymbol{u}'> = -\frac{\mu}{2}\left(\frac{1}{\mu}\right)', <\boldsymbol{u}' \times \boldsymbol{u}''', \boldsymbol{v}> = \sqrt{\mu} <\boldsymbol{u} \times \boldsymbol{u}', \boldsymbol{u}''>',$$

$$
\begin{aligned}
< \boldsymbol{v}, \boldsymbol{u}'' > &= -\sqrt{\mu} - \frac{\mu^2}{4}\left(\frac{1}{\mu}\right)'^2 + \frac{\mu}{2}\left(\frac{1}{\mu}\right)'', \\
< \boldsymbol{u}' \times \boldsymbol{u}'', \boldsymbol{v} > &= \mu < \boldsymbol{u} \times \boldsymbol{u}', \boldsymbol{u}'' > .
\end{aligned}
$$

Proof. The Christoffel symbols Γ^k_{ij} for the sphere $\mathbb{S}^3$ are given by

(2)
$$
\Gamma^k_{ij} = \Gamma^k_{ji} = \begin{cases} -\sqrt{\mu}.u^j, & k = i; \\ \sqrt{\mu}.u^k, & k \neq i = j; \\ 0, & k \neq i \neq j. \end{cases}
$$

Hence,

$$
\begin{aligned}
\sum_{i,j=1}^{3} \Gamma^1_{ij} u^{i'} u^{j'} &= \Gamma^1_{11}(u^{1'})^2 + 2\Gamma^1_{12} u^{1'} u^{2'} + 2\Gamma^1_{13} u^{1'} u^{3'} + \Gamma^1_{22}(u^{2'})^2 + \\
&+ \Gamma^1_{33}(u^{3'})^2 = -\sqrt{\mu} u^1 (u^{1'})^2 - 2\sqrt{\mu} u^2 u^{1'} u^{2'} - 2\sqrt{\mu} u^3 u^{1'} u^{3'} + \\
&+ \sqrt{\mu} u^1 (u^{2'})^2 + \sqrt{\mu} u^1 (u^{3'})^2 = -2\sqrt{\mu} u^{1'} (u^1 u^{1'} + u^2 u^{2'} + u^3 u^{3'}) + \\
&+ \sqrt{\mu} u^1 \underbrace{((u^{1'})^2 + (u^{2'})^2 + (u^{3'})^2)}_{=1} = -\sqrt{\mu}(\boldsymbol{u}^2)' u^{1'} + \sqrt{\mu} u^1 .
\end{aligned}
$$

Similarly, $\sum_{i,j=1}^{3} \Gamma^2_{ij} u^{i'} u^{j'} = -\sqrt{\mu}(\boldsymbol{u}^2)' u^{2'} + \sqrt{\mu} u^2$ and $\sum_{i,j=1}^{3} \Gamma^3_{ij} u^{i'} u^{j'} = -\sqrt{\mu}(\boldsymbol{u}^2)' u^{3'} + \sqrt{\mu} u^3$. Since $(\boldsymbol{u}^2)' = \|\boldsymbol{u}\|^{2'} = \left(\frac{2}{\sqrt{\mu}} - 1\right)' = \sqrt{\mu}\left(\frac{1}{\mu}\right)'$, we get the expression of the vector function $\boldsymbol{v}$. The next scalar products in the statement of the lemma are obtained by (1) and $< \boldsymbol{u}, \boldsymbol{u}' > = \frac{\sqrt{\mu}}{2}\left(\frac{1}{\mu}\right)'$ after the necessary differentiations. □

The proofs of the next lemmas are routine, replacing the Christoffel symbols by (2).

Lemma 2.2. *Let $\boldsymbol{w} = \boldsymbol{w}(s)$ be a vector function in $\mathbb{R}^3$ with coordinate functions $w^k(s) = \sum_{i,j=1}^{3} \Gamma_{ij}^k u^{i'} u^{j''}$, $k = 1,2,3$, where Γ_{ij}^k are the Christoffel symbols for the sphere $\mathbb{S}^3$ and $u^i = u^i(s)$, $i = 1,2,3$ are the coordinate functions of the unit speed curve $c : \boldsymbol{u} = \boldsymbol{u}(s)$. Then*

$$\boldsymbol{w} = -\frac{\mu}{2}\left(\frac{1}{\mu}\right)' .\boldsymbol{u}'' + \frac{1}{4}\left(4\sqrt{\mu} + \mu^2\left(\frac{1}{\mu}\right)'^2 - 2\mu\left(\frac{1}{\mu}\right)''\right).\boldsymbol{u}'$$

and

$$< \boldsymbol{w}, \boldsymbol{u}'' >= -\frac{\mu}{2}\left(\frac{1}{\mu}\right)' \varkappa^2, < \boldsymbol{w}, \boldsymbol{v} >= 0,$$
$$< \boldsymbol{w}, \boldsymbol{u}' >= - < \boldsymbol{v}, \boldsymbol{u}'' >= \sqrt{\mu} + \frac{\mu^2}{4}\left(\frac{1}{\mu}\right)'^2 - \frac{\mu}{2}\left(\frac{1}{\mu}\right)''.$$

Lemma 2.3. *Let $\eta = \eta(s)$ be a vector function in $\mathbb{R}^3$ with coordinate functions $\eta^k(s) = \sum_{i,j=1}^{3} \Gamma_{ij}^k u^{i'} (u' \times u'')^j$, $k = 1,2,3$, where Γ_{ij}^k are the Christoffel symbols for the sphere $\mathbb{S}^3$, $u^i = u^i(s)$, $i = 1,2,3$ are the coordinate functions of the unit speed curve $c : \boldsymbol{u} = \boldsymbol{u}(s)$ and $(u' \times u'')^j$, $j = 1,2,3$ are the coordinate functions of the vector product $\boldsymbol{u}' \times \boldsymbol{u}''$ in $\mathbb{R}^3$. Then*

$$\eta = -\frac{\mu}{2}\left(\frac{1}{\mu}\right)' .(\boldsymbol{u}' \times \boldsymbol{u}'') - \sqrt{\mu} < \boldsymbol{u} \times \boldsymbol{u}', \boldsymbol{u}'' > .\boldsymbol{u}'.$$

and

$$< \eta, \boldsymbol{u}'' >=< \eta, \boldsymbol{v} >= 0,$$
$$< \eta, \boldsymbol{u}' >= - < \boldsymbol{u}' \times \boldsymbol{u}'', \boldsymbol{v} >= -\sqrt{\mu} < \boldsymbol{u} \times \boldsymbol{u}', \boldsymbol{u}'' > .$$

Let c be a Frenet space curve in $\mathbb{R}^3$. We denote by γ the stereographic pre-image of c on the sphere $\mathbb{S}^3$, i.e. $\pi(\gamma) = c$. Let $\boldsymbol{l} = \boldsymbol{l}(\sigma)$, $\sigma \in J$ be the vector parametric equation of γ, parameterized by an arc-length parameter σ. We have that $\boldsymbol{l}^2 = 1$, $\boldsymbol{l}'^2 = 1$ and $<\boldsymbol{l},\boldsymbol{l}'>= 0$, $<\boldsymbol{l},\boldsymbol{l}''>= -1$, $<\boldsymbol{l},\boldsymbol{l}'''>= 0$. The vector field $\boldsymbol{l}+\boldsymbol{l}''$ is orthogonal to $\boldsymbol{l}$ and $(\boldsymbol{l}+\boldsymbol{l}'')^2 = \boldsymbol{l}''^2 - 1 \geqq 0$. A curve γ is said to be regular if $(\boldsymbol{l}+\boldsymbol{l}'')^2 = \boldsymbol{l}''^2 - 1 > 0$ for every $\sigma \in J$. A moving frame $\boldsymbol{ltnb}$ at any point of a regular curve γ on $\mathbb{S}^3$ is introduced as follows: $\boldsymbol{t} = \boldsymbol{l}'$, $\boldsymbol{n} = \dfrac{\boldsymbol{l}+\boldsymbol{l}''}{\|\boldsymbol{l}+\boldsymbol{l}''\|}$ and $\boldsymbol{b}$ is the unique vector, determined by the condition $\boldsymbol{ltnb}$ to be a right-handed orthonormal frame. The function $\widetilde{\varkappa} = \|\boldsymbol{l}+\boldsymbol{l}''\| = \sqrt{\boldsymbol{l}''^2 - 1} > 0$ is called a spherical curvature of γ. The vector field

$$\boldsymbol{n}' + \widetilde{\varkappa}\boldsymbol{t} = -\frac{\widetilde{\varkappa}'}{\widetilde{\varkappa}^2}(\boldsymbol{l}+\boldsymbol{l}'') + \frac{1+\widetilde{\varkappa}^2}{\widetilde{\varkappa}}\boldsymbol{l}' + \frac{1}{\widetilde{\varkappa}}\boldsymbol{l}'''$$

has the direction of the vector $\boldsymbol{b}$. So, we can put $\boldsymbol{n}' + \widetilde{\varkappa}\boldsymbol{t} = \widetilde{\tau}\boldsymbol{b}$. The function $\widetilde{\tau}$ is called a spherical torsion of γ. For the moving frame $\boldsymbol{ltnb}$ we obtain the following Frenet type formulas

$$\boldsymbol{l}' = \boldsymbol{t},\ \ \boldsymbol{t}' = -\boldsymbol{l} + \widetilde{\varkappa}\boldsymbol{n},\ \ \boldsymbol{n}' = -\widetilde{\varkappa}\boldsymbol{t} + \widetilde{\tau}\boldsymbol{b},\ \ \boldsymbol{b}' = -\widetilde{\tau}\boldsymbol{n}.$$

If we denote by $\nabla_{\boldsymbol{t}}$ the covariant differentiation along the curve γ on $\mathbb{S}^3$ then $\nabla_{\boldsymbol{t}}\boldsymbol{t} = \widetilde{\varkappa}\boldsymbol{n}$, $\nabla_{\boldsymbol{t}}\boldsymbol{n} = -\widetilde{\varkappa}\boldsymbol{t} + \widetilde{\tau}\boldsymbol{b}$ and $\nabla_{\boldsymbol{t}}\boldsymbol{b} = -\widetilde{\tau}\boldsymbol{n}$. Hence, $\widetilde{\varkappa} = \|\nabla_{\boldsymbol{t}}\boldsymbol{t}\|$ and $\widetilde{\tau} = - < \nabla_{\boldsymbol{t}}\boldsymbol{b}, \boldsymbol{n} >$ (see [6]).

3 Main theorem

The fundamental theorem in the space curve theory states that the Euclidean curvature $\varkappa$ and the Euclidean torsion τ determine the space curve, parameterized by an arc-length parameter, up to a rigid motion in $\mathbb{R}^3$. Also, the spherical curvature $\widetilde{\varkappa}$ and the spherical torsion $\widetilde{\tau}$ define the curve, lying on the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$, up to a rotation, preserving the sphere $\mathbb{S}^3$. Relations between the invariants $\varkappa, \tau, \widetilde{\varkappa}, \widetilde{\tau}$ of the corresponding curves via the stereographic projection π will define any Frenet space curve up to transformation of the group M_0.

Theorem 3.1. *Let c be a Frenet space curve in $\mathbb{R}^3$, parameterized by an arc-length parameter s, and let γ be its stereographic pre-image on* *the sphere $\mathbb{S}^3$, i.e. $\pi(\gamma) = c$. If $\varkappa = \varkappa(s)$ and $\tau = \tau(s)$ are the Euclidean curvature and the Euclidean torsion of the curve c and $\widetilde{\varkappa} = \widetilde{\varkappa}(s)$ and $\widetilde{\tau} = \widetilde{\tau}(s)$ are the spherical curvature and the spherical torsion of the curve γ then*

$$\widetilde{\varkappa}^2 = \frac{1}{\mu}\varkappa^2 - 1 - \frac{3\mu}{4}\left(\frac{1}{\mu}\right)'^2 + \left(\frac{1}{\mu}\right)'' \tag{3}$$

$$\widetilde{\tau} = \frac{\cos\theta}{\mu\widetilde{\varkappa}}\varkappa\tau - \frac{\sin\theta}{\mu\widetilde{\varkappa}}\varkappa', \tag{4}$$

where

$$\cos\theta = \frac{1}{\sqrt{\mu}\varkappa\widetilde{\varkappa}}\left(\varkappa^2 - \frac{1}{4}\mu^2\left(\frac{1}{\mu}\right)'^2 + \frac{1}{2}\mu\left(\frac{1}{\mu}\right)'' - \sqrt{\mu}\right). \tag{5}$$

Proof. Let $\boldsymbol{l} = \boldsymbol{l}(s)$ be a vector parametric equation of γ and $\sigma = \sigma(s)$ be its arc-length function. If $c : \boldsymbol{u}(s) = (u^1(s), u^2(s), u^3(s))$ then $\dot{\boldsymbol{l}} = \dfrac{d\boldsymbol{l}}{ds} = \sum_{i=1}^{3}(u^i)'\boldsymbol{l}_i$. The form $d\sigma = \sqrt{\mu}ds$ is the linear element of the curve γ on the sphere $\mathbb{S}^3$. Thus, the unit tangent vector field is represented by $\boldsymbol{t} = \dfrac{1}{\sqrt{\mu}}\dot{\boldsymbol{l}}$. Applying the covariant differentiation $\nabla_{\boldsymbol{t}}$ we obtain that

$$\nabla_{\boldsymbol{t}}\boldsymbol{t} = \sum_{k=1}^{3}\left(\frac{1}{\mu}(u^k)'' + \frac{1}{\mu}v^k + \frac{1}{2}\left(\frac{1}{\mu}\right)'(u^k)'\right)\boldsymbol{l}_k, \tag{6}$$

where $v^k = v^k(s)$, $k = 1,2,3$ are the coordinate functions of the vector function $\boldsymbol{v}$, defined in Lemma 2.1. Hence, $\widetilde{\varkappa}^2 =$

$$= \|\nabla_{\boldsymbol{t}}\boldsymbol{t}\|^2 = \frac{1}{\mu}\varkappa^2 + \frac{1}{\mu}\boldsymbol{v}^2 + \frac{\mu}{4}\left(\frac{1}{\mu}\right)'^2 + \frac{2}{\mu} < \boldsymbol{u}'', \boldsymbol{v} > + \left(\frac{1}{\mu}\right)' < \boldsymbol{v}, \boldsymbol{u}' >$$

and replacing with the equalities in Lemma 2.1 we get (3). The stereographic projection π and the reverse map π^{-1} are conformal maps and the differential π_*^{-1} of the reverse map π^{-1} is a similarity transformation between the corresponding tangent spaces with a coefficient $\sqrt{\mu}$. If $\boldsymbol{T}=\boldsymbol{u}'$, $\boldsymbol{N}=\dfrac{\boldsymbol{u}''}{\varkappa}$, $\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}=\dfrac{\boldsymbol{u}'\times\boldsymbol{u}''}{\varkappa}$ are the Frenet frame vector field along the curve c then $\pi_*^{-1}(\boldsymbol{N})=\dfrac{1}{\varkappa}\sum\limits_{i=1}^{3}(u^i)''\boldsymbol{l}_i$ and $\pi_*^{-1}(\boldsymbol{B})=\dfrac{1}{\varkappa}\sum\limits_{i=1}^{3}(u'\times u'')^i\boldsymbol{l}_i$. Vectors $\pi_*^{-1}(\boldsymbol{N})$ and $\pi_*^{-1}(\boldsymbol{B})$, lying in the plane, determined by vectors $\boldsymbol{n}$ and $\boldsymbol{b}$, are orthogonal and let $\theta=\angle(\boldsymbol{n},\pi_*^{-1}(\boldsymbol{N}))$. Then $\boldsymbol{n}=\dfrac{\cos\theta}{\sqrt{\mu}}\pi_*^{-1}(\boldsymbol{N})+\dfrac{\sin\theta}{\sqrt{\mu}}\pi_*^{-1}(\boldsymbol{B})$. Hence,

$$\cos\theta=\frac{1}{\sqrt{\mu}}<\boldsymbol{n},\pi_*^{-1}(\boldsymbol{N})>=\frac{1}{\widetilde{\varkappa}\varkappa\sqrt{\mu}}<\nabla_{\boldsymbol{t}}\boldsymbol{t},\sum_{i=1}^{3}(u^i)''\boldsymbol{l}_i>$$

and

$$\sin\theta=\frac{1}{\sqrt{\mu}}<\boldsymbol{n},\pi_*^{-1}(\boldsymbol{B})>=\frac{1}{\widetilde{\varkappa}\varkappa\sqrt{\mu}}<\nabla_{\boldsymbol{t}}\boldsymbol{t},\sum_{i=1}^{3}(u'\times u'')^i\boldsymbol{l}_i>.$$

Using (6) we obtain that $\cos\theta=\dfrac{1}{\widetilde{\varkappa}\varkappa\sqrt{\mu}}\left(\boldsymbol{u}''^2+<\boldsymbol{v},\boldsymbol{u}''>\right)=$
$=\dfrac{1}{\widetilde{\varkappa}\varkappa\sqrt{\mu}}\left(\varkappa^2+<\boldsymbol{v},\boldsymbol{u}''>\right)$, whence applying Lemma 2.1 we get (5). Also,

$$\sin\theta=\frac{1}{\widetilde{\varkappa}\varkappa\sqrt{\mu}}<\boldsymbol{v},\boldsymbol{u}'\times\boldsymbol{u}''>=\frac{1}{\widetilde{\varkappa}\varkappa}<\boldsymbol{u}\times\boldsymbol{u}',\boldsymbol{u}''>. \tag{7}$$

Observing that π is an orientation preserving map in odd dimension, we have that $\boldsymbol{b}=\dfrac{-\sin\theta}{\sqrt{\mu}}\pi_*^{-1}(\boldsymbol{N})+\dfrac{\cos\theta}{\sqrt{\mu}}\pi_*^{-1}(\boldsymbol{B})=$

$= -\dfrac{1}{\varkappa\sqrt{\mu}}\sum_{i=1}^{3}\left(\sin\theta(u^i)'' - \cos\theta(u'\times u'')^i\right)\boldsymbol{l}_i$. Therefore, using the covariant differentiation technique, we get

$$\nabla_{\boldsymbol{t}}\boldsymbol{b} = \frac{1}{\sqrt{\mu}}\nabla_{\boldsymbol{i}}\boldsymbol{b} = -\frac{1}{\sqrt{\mu}}\sum_{k=1}^{3}\left[\frac{1}{\varkappa\sqrt{\mu}}\Big(\sin\theta(w^k + (u^k)''') -\right.$$

$$\left. -\cos\theta(\eta^k + (u'\times u''')^k)\Big) + \left(\frac{\sin\theta}{\varkappa\sqrt{\mu}}\right)'(u^k)'' - \left(\frac{\cos\theta}{\varkappa\sqrt{\mu}}\right)'(u'\times u'')^k\right]\boldsymbol{l}_k.$$

Thus,

$$\widetilde{\tau} = -<\nabla_{\boldsymbol{t}}\boldsymbol{b}, n> = -\frac{1}{\widetilde{\varkappa}} <\nabla_{\boldsymbol{t}}\boldsymbol{b}, \nabla_{\boldsymbol{t}}\boldsymbol{t}> = \frac{\sqrt{\mu}}{\widetilde{\varkappa}}\left[\frac{\sin\theta}{\varkappa\mu\sqrt{\mu}}\cdot\right.$$

$$\Bigg(\underbrace{<\boldsymbol{w},\boldsymbol{u}''> + <\boldsymbol{u}''',\boldsymbol{u}''> + <\boldsymbol{w},\boldsymbol{v}> + <\boldsymbol{u}''',\boldsymbol{v}>}_{\text{A}}\Bigg) + \frac{\sin\theta}{2\varkappa\sqrt{\mu}}\left(\frac{1}{\mu}\right)'\cdot$$

$$\Bigg(\underbrace{<\boldsymbol{w},\boldsymbol{u}'> + <\boldsymbol{u}''',\boldsymbol{u}'>}_{\text{B}}\Bigg) - \frac{\cos\theta}{\varkappa\mu\sqrt{\mu}}\cdot$$

$$\Bigg(\underbrace{<\eta,\boldsymbol{u}''> + <\boldsymbol{u}'\times\boldsymbol{u}''',\boldsymbol{u}''> + <\eta,\boldsymbol{v}> + <\boldsymbol{u}'\times\boldsymbol{u}''',\boldsymbol{v}>}_{\text{C}}\Bigg) -$$

$$-\frac{\cos\theta}{2\varkappa\sqrt{\mu}}\left(\frac{1}{\mu}\right)'\underbrace{<\eta,\boldsymbol{u}'>}_{\text{D}} +$$

$$\left. +\frac{1}{\mu}\left(\frac{\sin\theta}{\varkappa\sqrt{\mu}}\right)'\Bigg(\underbrace{\varkappa^2 + <\boldsymbol{u}'',\boldsymbol{v}>}_{\text{E}}\Bigg) - \frac{1}{\mu}\left(\frac{\cos\theta}{\varkappa\sqrt{\mu}}\right)'\underbrace{<\boldsymbol{u}'\times\boldsymbol{u}'',\boldsymbol{v}>}_{\text{F}}\right] =$$

$$= \frac{1}{\widetilde{\varkappa}}\left[\frac{\sin\theta}{\varkappa\mu}.A + \frac{\sin\theta}{2\varkappa}\left(\frac{1}{\mu}\right)'.B - \frac{\cos\theta}{\varkappa\mu}.C - \frac{\cos\theta}{2\varkappa}\left(\frac{1}{\mu}\right)'.D + \right.$$
$$\left. + \frac{1}{\sqrt{\mu}}\left(\frac{\sin\theta}{\varkappa\sqrt{\mu}}\right)'.E - \frac{1}{\sqrt{\mu}}\left(\frac{\cos\theta}{\varkappa\sqrt{\mu}}\right)'.F.\right]$$

Applying the lemmas 2.1, 2.2, 2.3 and taking account of the equalities (1), (5), (7) we have that $A = -\varkappa\varkappa' + \sqrt{\mu}(\varkappa\widetilde{\varkappa}\cos\theta)'$, $-B = E = \sqrt{\mu}\varkappa\widetilde{\varkappa}\cos\theta, C = -\varkappa^2\tau + \sqrt{\mu}(\varkappa\widetilde{\varkappa}\sin\theta)'$, $-D = F = \sqrt{\mu}\varkappa\widetilde{\varkappa}\sin\theta$. Replacing the found expressions in the equality above we obtain (4) after simple rearranges and simplifications. □

4 Möbius invariants of space curves

It is obvious that the arc-length function $\sigma = \sigma(s)$ is an invariant with respect to the group M_0. Let we denote by $S_\sigma(s)$ the Schwarzian derivative of the function $\sigma = \sigma(s)$. Then we have that

$$S_\sigma(s) = \left(\frac{\sigma''}{\sigma'}\right)' - \frac{1}{2}\left(\frac{\sigma''}{\sigma'}\right)^2 = \frac{1}{2}\left[\frac{3}{4}\mu^2\left(\frac{1}{\mu}\right)'^2 - \mu\left(\frac{1}{\mu}\right)''\right].$$

If $s = s(\sigma)$ is the reverse function of σ then we get $0 = S_{\sigma\circ s}(\sigma) = S_\sigma(s)\left(\frac{ds}{d\sigma}\right)^2 + S_s(\sigma) \Rightarrow S_\sigma(s) = -\mu S_s(\sigma)$ applying the chain rule for the Schwarzian derivatives. The functions $\widetilde{\varkappa}$ and $\widetilde{\tau}$ from Theorem 3.1 can be expressed in terms of the arc-length parameter σ. We set $\mathfrak{K}(\sigma) = \widetilde{\varkappa}^2(\sigma)$, $\mathfrak{T}(\sigma) = \widetilde{\tau}(\sigma)$ and we obtain that

$$\text{(8)} \qquad \begin{aligned} \mathfrak{K}(\sigma) &= \frac{1}{\mu}\varkappa^2 - 1 + 2S_s(\sigma) \\ \mathfrak{T}(\sigma) &= \frac{1}{\mu\sqrt{\mathfrak{K}}}\left(\varkappa\tau\cos\theta - \sqrt{\mu}\frac{d\varkappa}{d\sigma}\sin\theta\right), \end{aligned}$$

where $\cos\theta = \dfrac{1}{\varkappa\sqrt{\mu\mathfrak{K}}}\left(\varkappa^2 + \mu S_s(\sigma) + \dfrac{1}{8\mu}\left(\dfrac{d\mu}{d\sigma}\right)^2 - \sqrt{\mu}\right).$

Corollary 4.1. *The functions $\mathfrak{K} = \mathfrak{K}(\sigma)$ and $\mathfrak{T} = \mathfrak{T}(\sigma)$, define by (8), are invariants under the group M_0.*

Proof. The proof is omitted. $\square$

Theorem 4.2 (Uniqueness theorem)**.** *Let $I \subset \mathbb{R}$ be an open interval and let $c_i : I \to \mathbb{R}^3$, $i = 1,2$ be two Frenet space curves, parameterized by the same arc-length parameter σ of their stereographic pre-images γ_i, $i = 1,2$, respectively, on the sphere $\mathbb{S}^3$ in $\mathbb{R}^4$. Assume that the curves c_1 and c_2 have the same invariants $\mathfrak{K}_i = \mathfrak{K}_i(\sigma)$, $\mathfrak{T}_i = \mathfrak{T}_i(\sigma)$, $i = 1,2$, defined by (8) for any $\sigma \in I$. Then there exists a transformation $F \in M_0$ such that $c_2 = F(c_1)$.*

Proof. Since $\mathfrak{K}_1 = \mathfrak{K}_2$ and $\mathfrak{T}_1 = \mathfrak{T}_2$ then the spherical stereographic pre-images γ_1 and γ_2 of the curves c_1 and c_2, respectively, via the stereographic projection π, have the same spherical curvatures and torsions. Therefore, there exists a rotation $\rho \in SO(4)$ such that $\gamma_2 = \rho(\gamma_1)$. Hence, $c_2 = \pi(c_1) = \pi \circ \rho(\gamma_1) = \pi \circ \rho \circ \pi^{-1}(c_1)$, where $F = \pi \circ \rho \circ \pi^{-1} \in M_0$ and the proof is completed. $\square$

Theorem 4.3 (Existence theorem)**.** *Let $f : I \to \mathbb{R}$, $f > 0$ and $g : I \to \mathbb{R}$ be given C^∞–functions, defined on the same interval $I \subset \mathbb{R}$. Let $\boldsymbol{c}_0 \in \mathbb{R}^3$ and $\boldsymbol{e}_1^0$, $\boldsymbol{e}_2^0$, $\boldsymbol{e}_3^0$ be a right-handed orthonormal frame at $\boldsymbol{c}_0$ in the Euclidean space $\mathbb{R}^3$. There exists a unique Frenet space curve $c : I \to \mathbb{R}^3$ which satisfies the conditions:*

(a) there exists $\sigma_0 \in I$, such that $c(\sigma_0) = \boldsymbol{c}_0$, and the Frenet frame of c at $\boldsymbol{c}_0$ is $\boldsymbol{e}_1^0$, $\boldsymbol{e}_2^0$, $\boldsymbol{e}_3^0$;

(b) for any $\sigma \in I$ $\mathfrak{K}(\sigma) = f^2(\sigma)$ and $\mathfrak{T}(\sigma) = g(\sigma)$.

Proof. Let $\boldsymbol{l}_0 = \pi^{-1}(\boldsymbol{c}_0)$ and $\boldsymbol{t}_0 = \pi_*^{-1}(\boldsymbol{e}_1^0)$, $\boldsymbol{n}_0 = \pi_*^{-1}(\boldsymbol{e}_2^0)$, $\boldsymbol{b}_0 = \pi_*^{-1}(\boldsymbol{e}_3^0)$. Let us consider a matrix-valued function
$\mathscr{E}(\sigma) = (\boldsymbol{l}(\sigma), \boldsymbol{t}(\sigma), \boldsymbol{n}(\sigma), \boldsymbol{b}(\sigma))^T$. Solving the system of first order linear differential equations

$\dfrac{d}{d\sigma}\mathscr{E} = \mathscr{A}(\sigma)\mathscr{E}$, with a given matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & f & 0 \\ 0 & -f & 0 & g \\ 0 & 0 & -g & 0 \end{pmatrix}$ and initial conditions $\boldsymbol{l}_0$, $\boldsymbol{t}_0$, $\boldsymbol{n}_0$, $\boldsymbol{b}_0$ we obtain a unique solution $\mathscr{E} = \mathscr{E}(\sigma)$, determined for all $\sigma \in I$ and $\mathscr{E}(\sigma_0) = (\boldsymbol{l}_0, \boldsymbol{t}_0, \boldsymbol{n}_0, \boldsymbol{b}_0)^T$ for some $\sigma_0 \in I$. It is routine to prove that the matrix $\mathscr{E}$ is orthogonal. This means that the vectors $\boldsymbol{l}(\sigma)$, $\boldsymbol{t}(\sigma)$, $\boldsymbol{n}(\sigma)$, $\boldsymbol{b}(\sigma)$ form an orthogonal frame in $\mathbb{R}^4$ for any $\sigma \in I$. Let γ be a spherical curve, defined by the vector function $\boldsymbol{l} = \boldsymbol{l}(\sigma)$ and let $c = \pi(\gamma)$. It is clear that the conditions (a) and (b) in the statement of the theorem are fulfilled for the space curve c. $\square$

The proof of the last theorem give us an algorithm of recovering space curves up to a transformation from the group M_0.

Algorithm. *Recovering space curves by two functions $f = f(\sigma) > 0$ and $g = g(\sigma)$ for any $\sigma \in I$.*

1. *Choose initial conditions:* $\boldsymbol{c}_0$, $\boldsymbol{e}_1^0$, $\boldsymbol{e}_2^0$, $\boldsymbol{e}_3^0$;

2. *Find the vectors* $\boldsymbol{l}_0 = \pi^{-1}(\boldsymbol{c}_0)$, $\boldsymbol{t}_0 = \pi_*^{-1}(\boldsymbol{e}_1^0)$, $\boldsymbol{n}_0 = \pi_*^{-1}(\boldsymbol{e}_2^0)$, $\boldsymbol{b}_0 = \pi_*^{-1}(\boldsymbol{e}_3^0)$;

3. *Solve the differential equation* $\dfrac{d}{d\sigma}\mathscr{E} = \mathscr{A}(\sigma)\mathscr{E}$, *where* $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & f & 0 \\ 0 & -f & 0 & g \\ 0 & 0 & -g & 0 \end{pmatrix}$, *and initial conditions, determined in Step 1., for* $\mathscr{E}(\sigma_0) = (\boldsymbol{l}_0, \boldsymbol{t}_0, \boldsymbol{n}_0, \boldsymbol{b}_0)^T$;

4. *The curves γ, with vector function $\boldsymbol{l} = \boldsymbol{l}(\sigma)$, and $c = \pi(\gamma)$ are found.*

The next two examples illustrate the considered conformal invariants of a space curve in $\mathbb{R}^3$. For the calculations and visualizations we use the computer system Mathematica.

Example 4.4. *Let*

$$c : \boldsymbol{u}(\sigma) = \left\{\cos\left(\sqrt{2}\tan\left(\frac{\sigma}{2}\right)\right), \sin\left(\sqrt{2}\tan\left(\frac{\sigma}{2}\right)\right), \sqrt{2}\tan\left(\frac{\sigma}{2}\right)\right\}$$

be a helix in $\mathbb{R}^3$, *parameterized by an arc-length parameter* σ *of its spherical stereographic pre-image. Then* $\mathfrak{K}(\sigma) = \dfrac{1}{4\cos^4\left(\frac{\sigma}{2}\right)}$ *and* $\mathfrak{T}(\sigma) = \dfrac{\cos\sigma}{1+\cos\sigma}$.

Example 4.5. *Let* $\mathfrak{K}(\sigma) = \sigma^2$, $\mathfrak{T}(\sigma) = 0.6\sigma$, $\boldsymbol{c}_0 = (0,0,0)$, $\boldsymbol{e}_1^0 = (0,0,1)$, $\boldsymbol{e}_2^0 = (0,1,0)$. *Then, applying the algorithm above, where* $f(\sigma) = \sigma$, *and* $g(\sigma) = 0.6\sigma$, *we obtain the Frenet space curve, depicted in Fig.1.*

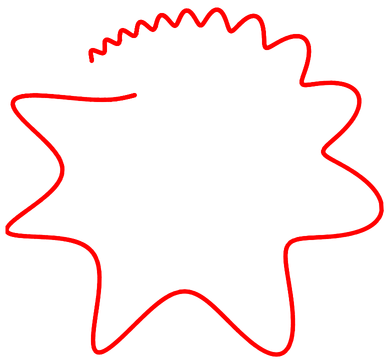

Fig. 1: $\mathfrak{K}(\sigma) = \sigma^2$, $\mathfrak{T}(\sigma) = 0.6\sigma$

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