

THE INVERSE WEIERSTRASS ITERATIVE METHOD AS A PROJECTION METHOD FOR SOLVING EIGENVALUE PROBLEM*

GYURHAN H. NEDZHIBOV

ABSTRACT: *In this work we present a new method for solving the eigenvalue problem of a companion matrix. We show that the introduced method is equivalent to the Inverse Weierstrass iterative method for simultaneous extraction of all polynomial roots. Based on this dependence, we get some new theoretical properties for the Inverse Weierstrass method.*

KEYWORDS: *Companion matrix, Eigenvalue problem, Projection methods, Polynomial root-finding methods, Inverse Weierstrass method*

1 Introduction

We consider the particular eigenvalue problem: find a scalar $\lambda \in \mathbb{C}$ and a non-zero vector $\mathbf{x} \in \mathbb{C}^n$ such that

$$(1) \quad A\mathbf{x} = \lambda\mathbf{x},$$

where A is the following (*inverse*) *companion matrix* of order n with complex entries and with only simple, and non-zero eigenvalues

$$(2) \quad A = \begin{pmatrix} -\frac{a_1}{a_0} & -\frac{a_2}{a_0} & \dots & -\frac{a_{n-1}}{a_0} & -\frac{1}{a_0} \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \text{ where } a_0 \neq 0.$$

Let us denote by $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}$ the two by two distinct and non-zero eigenvalues of A . Then it is immediate to verify that the eigenvector associated with $\frac{1}{\lambda_s}$ has the form

$$\mathbf{x}_s = (1, \lambda_s, \lambda_s^2, \dots, \lambda_s^{n-1})^T,$$

for $s = 1, \dots, n$. Therefore the eigenvector matrix V of A is the well-known *Vandermonde matrix*

$$(3) \quad V = V(\lambda) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \dots & \dots & \ddots & \dots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{pmatrix}.$$

Thus we have the following eigenvalue decomposition of (2)

$$(4) \quad A = V\Lambda V^{-1},$$

where $\Lambda = \text{diag}\{\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}\}$.

*This work is supported by Shumen University under Grant RD-08-112/31.01.2020

2 Projection methods for eigenvalue problems

In this section we give a short introduction to the Projection methods for solving the eigenvalue problem. Given two m -dimensional subspaces \mathcal{K} and \mathcal{L} of \mathbf{C}^n , the basic idea of projection methods is to approximate the exact eigenpair (λ, \mathbf{x}) to the eigenvalue problem (1), by an approximate eigenpair $(\hat{\lambda}, \hat{\mathbf{x}})$, with $\hat{\lambda} \in \mathbf{C}$ and $\hat{\mathbf{x}} \in \mathcal{K}$, such that the following *Petrov-Galerkin condition* is satisfied

$$(5) \quad A\hat{\mathbf{x}} - \hat{\lambda}\hat{\mathbf{x}} \perp \mathcal{L}$$

for some given inner product. The subspace \mathcal{K} referred to as the subspace of approximants or the *right subspace* and \mathcal{L} as the *left subspace*.

There are two main classes of projection methods: *orthogonal projection* (Ritz-Galerkin) and *oblique projection* methods (see for more details [1, 2, 4, 5]). In the case of orthogonal projection methods the subspace \mathcal{K} is the same as \mathcal{L} , while in oblique projection techniques subspace \mathcal{K} is different from \mathcal{L} .

We can translate in matrix form the approximate eigenvector $\hat{\mathbf{x}}$ in some basis and express the Petrov-Galerkin condition. Let assume that we have a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ of \mathcal{K} and denote by V the matrix with column vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$. By analogy, let $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ be a basis of \mathcal{L} and denote by W the matrix with column vectors $\mathbf{w}_1, \dots, \mathbf{w}_m$. Usually, the following additional assumption is required

$$(6) \quad W^*V = I,$$

which means that the two bases are biorthogonal, i.e. $(\mathbf{v}_i, \mathbf{w}_j) = \delta_{ij}$. Now we can translate the approximate problem into this basis and solve it numerically. Letting

$$\hat{\mathbf{x}} = V\mathbf{u}$$

the condition (5) becomes

$$\langle (A - \hat{\lambda}I)V\mathbf{u}, \mathbf{w}_j \rangle = 0, \quad j = 1, \dots, m.$$

Therefore, \mathbf{u} and $\hat{\lambda}$ must satisfy

$$(7) \quad B_m\mathbf{u} = \hat{\lambda}\mathbf{u},$$

where

$$(8) \quad B_m = W^*AV.$$

In this way, we reduced the initial eigenvalue problem to the smaller $m \times m$ eigenvalue problem. Each eigenvalue $\hat{\lambda}$ of B_m is called a *Ritz value*, and $V\mathbf{u}$ is called *Ritz vector*, where \mathbf{u} is the eigenvector of B_m associated with $\hat{\lambda}$.

In the case of orthogonal projection methods the matrix B_m has the form $B_m = V^*AV$. In this case, the method for solving the reduced eigenvalue problem (7) is also known as the *Rayleigh-Ritz procedure*.

We can also reformulate projection methods in terms of projection operators as follows.

Case 1. Orthogonal projection methods. Let $P_{\mathcal{K}}$ be the orthogonal projector onto \mathcal{K} , then condition (5) has the form

$$(9) \quad A\hat{\mathbf{x}} - \hat{\lambda}\hat{\mathbf{x}} \perp \mathcal{K}$$

or equivalently,

$$P_{\mathcal{K}}(A - \hat{\lambda}I)\hat{\mathbf{x}} = 0, \quad (\hat{\lambda} \in \mathbf{C}, \hat{\mathbf{x}} \in \mathcal{K})$$

or

$$(10) \quad P_{\mathcal{K}}A\hat{\mathbf{x}} = \hat{\lambda}\hat{\mathbf{x}}.$$

Hence, we have replaced the initial problem (1) by an eigenvalue problem for the linear transformation $P_{\mathcal{K}}A|_{\mathcal{K}}$ which is from \mathcal{K} to \mathcal{K} . The last equation is equivalent to

$$(11) \quad P_{\mathcal{K}}AP_{\mathcal{K}}\hat{\mathbf{x}} = \hat{\lambda}\hat{\mathbf{x}}, \quad (\hat{\lambda} \in \mathbf{C}, \hat{\mathbf{x}} \in \mathcal{K}),$$

which involves the operator

$$(12) \quad A_m = P_{\mathcal{K}}AP_{\mathcal{K}}.$$

This is an extension of $P_{\mathcal{K}}A|_{\mathcal{K}}$ to the whole space. Equation (10) will be referred to as the Galerkin approximation problem.

Case 2. Oblique projection methods. Let $Q_{\mathcal{K}}^{\mathcal{L}}$ be the oblique projector onto \mathcal{K} and orthogonal to \mathcal{L} . This projector has the following properties:

$$(13) \quad Q_{\mathcal{K}}^{\mathcal{L}}\mathbf{z} \in \mathcal{K} \quad \text{and} \quad (I - Q_{\mathcal{K}}^{\mathcal{L}})\mathbf{z} \perp \mathcal{L}.$$

for any vector $\mathbf{z} \in \mathbf{C}^n$. Then the orthogonality condition (5) is equivalent to

$$Q_{\mathcal{K}}^{\mathcal{L}}(A - \hat{\lambda}I)\hat{\mathbf{x}} = 0, \quad (\hat{\lambda} \in \mathbf{C}, \hat{\mathbf{x}} \in \mathcal{K})$$

or

$$(14) \quad Q_{\mathcal{K}}^{\mathcal{L}}A\hat{\mathbf{x}} = \hat{\lambda}\hat{\mathbf{x}}.$$

Thus, the eigenvalue problem for matrix A is replaced by the eigenvalue problem of the linear operator $Q_{\mathcal{K}}^{\mathcal{L}}A|_{\mathcal{K}}$. In this case, we can define an extension A_m by analogy to the definition of (12) in two different ways. One possible presentation could be

$$(15) \quad A_m = Q_{\mathcal{K}}^{\mathcal{L}}AQ_{\mathcal{K}}^{\mathcal{L}}$$

or, the more useful extension

$$(16) \quad A_m = Q_{\mathcal{K}}^{\mathcal{L}}AP_{\mathcal{K}}.$$

Then by analogy to (11) we get the approximation problem

$$(17) \quad A_m\hat{\mathbf{x}} = \hat{\lambda}\hat{\mathbf{x}}, \quad (\hat{\lambda} \in \mathbf{C}, \hat{\mathbf{x}} \in \mathcal{K}).$$

Therefore in both cases (11) and (17), the Ritz values are actually the eigenvalues of the operator A_m , and the Ritz vectors are the corresponding eigenvectors belonging to \mathcal{H} .

There are different ways to construct the subspaces \mathcal{H} and \mathcal{L} . We can work with subspaces of both fixed and variable dimensions. Some of the well-known examples when the subspace dimension is fixed to one are the *Power method* and *Rayleigh quotient iteration* (see [2, 9]). In other projection methods the dimension of the subspace \mathcal{H} is increasing, usually one starts with a subspace of dimension one, and the dimension increases by one at each iteration step. Some of the most popular methods of this type use the so-called *Krylov subspace*. Such examples are Arnoldi method, Lanczos method and incomplete orthogonalization (see [1, 3]). Methods like *Davidson* and *Jacobi-Davidson method* are with increasing subspace dimension and without using Krylov subspaces (see [6, 7]).

3 A new iterative scheme for diagonalization of matrix (2)

We introduce the following new iterative scheme for finding all the eigenvalues and corresponding left, and right eigenvectors of the inverse companion matrix (2).

Algorithm-1: Diagonalization of the inverse companion matrix

Input: A , initial vector $\mathbf{z}^{(0)}$ (where $z_i^{(0)} \neq z_j^{(0)}$ for $i \neq j$), tolerance $\varepsilon \ll 1$.

Output: Approximate eigenvalues and eigenvector matrices:

$$\lambda = (\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}), V(\lambda), W(\lambda).$$

1: Set $k=0$.

2: **While** not converged **do**

3: Compute the Vandermonde matrix $V_k = V(\mathbf{z}^{(k)})$.

4: Compute the inverse Vandermonde matrix $W_k^* = V(\mathbf{z}^{(k)})^{-1}$.

5: **For** $i=1:n$ **do**

6: Compute next eigenvalue estimate

$$z_i^{(k+1)} = \rho_k = (\mathbf{w}_i^{(k)})^* A \mathbf{v}_i^{(k)},$$

where $\mathbf{v}_i^{(k)}$ is the i -th column vector of V_k ,

and $(\mathbf{w}_i^{(k)})^*$ is the i -th row vector of W_k^*

7: **End for**

8: **Set** $k=k+1$.

8: **If** $\|\mathbf{z}^{(k+1)} - \mathbf{z}^{(k)}\|_2 < \varepsilon$ **then**

9: **Set** $\lambda = \mathbf{z}^{(k+1)}$, $V = V_k$, $W = W_k^*$.

10: **break**

11: **End If**

12: **End While**

Where $z_i^{(k)}$ is the k -th approximation of the eigenvalue $\frac{1}{\lambda_i}$ ($i = 1, \dots, n$). The right and left approximate eigenvectors associated with eigenvalue $\frac{1}{\lambda_i}$ are denoted by $\mathbf{v}_i^{(k)}$ and $\mathbf{w}_i^{(k)}$, respectively. On step 6 of *Algorithm-1*, the two-sided *Rayleigh quotient*

$$(18) \quad \rho(\mathbf{x}, \mathbf{y}) = \rho(A, \mathbf{x}, \mathbf{y}) = \frac{\mathbf{y}^* A \mathbf{x}}{\mathbf{y}^* \mathbf{x}}$$

is used for computing the approximate eigenvalue. Formula (18) is introduced for the first time by Ostrowski in [8] (see also [9, 10]).

We will show that the presented *Algorithm-1* can be formulated in terms of projection methods. Without loss of generality we fix the value of index i on step 6 of *Algorithm-1*. Let denote by

$$(19) \quad \mathcal{H}_i = \text{span}\{\mathbf{v}_i^{(k)}\}$$

the right subspace and by

$$(20) \quad \mathcal{L}_i = \text{span}\{(\mathbf{w}_i^{(k)})^*\}$$

the left subspace associated with $\frac{1}{\lambda_i}$ at iteration step k . It is obvious that the biorthogonality condition (6) is valid. Then the corresponding reduced eigenvalue problem (7)-(8) of (1) has the following trivial form

$$(\mathbf{w}_i^{(k)})^* A \mathbf{v}_i^{(k)} \mathbf{y} = \hat{\lambda}_i \mathbf{y},$$

where the matrix B_m is a scalar ($m = 1$) and it is identical to the Rayleigh quotient (18). In other words, we unambiguously obtain the Ritz value

$$\hat{\lambda}_i = z_i^{(k+1)} = \rho(\mathbf{v}_i^{(k)}, \mathbf{w}_i^{(k)}).$$

which is the next approximation of $\frac{1}{\lambda_i}$. Furthermore, it is immediate to verify that the residual vector of approximate eigenpair $\left(\frac{1}{\hat{\lambda}_i}, \mathbf{v}_i^{(k)}\right)$

$$\mathbf{r}_i^{(k)} = A \mathbf{v}_i^{(k)} - \frac{1}{\hat{\lambda}_i} \mathbf{v}_i^{(k)}$$

is orthogonal to \mathcal{L}_i , i.e. the Petrov-Galerkin condition (5) is satisfied. With this, we confirmed that the approach of *Algorithm-1* is an oblique projection method with subspaces \mathcal{H}_i and \mathcal{L}_i having dimensions fixed to one.

4 Equivalence of Inverse Weierstrass method and introduced Algorithm-1

In this section we use the same notations as in the previous two sections. Let $P(z)$ be a monic polynomial of degree $n \geq 2$

$$(21) \quad P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0,$$

where $a_0 \neq 0$ and with n distinct zeros. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be pairwise distinct roots of $P(z)$.

In our previous work [11] we suggested the Inverse Weierstrass iterative method defined by the sequence

$$(22) \quad \mathbf{z}^{(k+1)} = \phi(\mathbf{z}^{(k)}), k = 0, 1, 2, \dots$$

where $\phi : \mathcal{D} \subset \mathbf{C}^n \rightarrow \mathbf{C}^n$ is a vector valued function with components

$$(23) \quad \phi_i(\mathbf{z}^{(k)}) = \frac{z_i^{(k)}}{1 - \frac{P(z_i^{(k)})}{a_0} \prod_{j \neq i}^n \frac{z_j^{(k)}}{z_j^{(k)} - z_i^{(k)}}, (i = 1, \dots, n)$$

and \mathcal{D} is the set of all vectors in \mathbf{C}^n with distinct components.

In [11] it is also shown that the entries of the matrix

$$(24) \quad H(\mathbf{z}) = V^{-1}(\mathbf{z})AV(\mathbf{z}),$$

where $\mathbf{z} = (z_1, \dots, z_n)$ and the matrices A and $V(\mathbf{z})$ are defined by (2) and (3), respectively, have the form

$$(25) \quad H_{ii} = \frac{1 - \frac{P(z_i)}{a_0} \prod_{j \neq i}^n \frac{z_j}{z_j - z_i}}{z_i} \quad \text{and} \quad H_{ij} = -\frac{P(z_j)}{a_0} \frac{1}{z_i} \prod_{j \neq i}^n \frac{z_j}{z_j - z_i}, \quad (i, j = 1, \dots, n).$$

Let now consider again *Algorithm-1*, step 6. The two-sided *Rayleigh quotient* computed on this step is exactly the diagonal element H_{ii} of (24), which approximates the ratio $\frac{1}{\lambda_i}$. From this and from expression (22) we conclude that the Inverse Weierstrass method is equivalent to introduced *Algorithm-1*.

5 Convergence analysis

Our main purpose in this section is to apply some of the known convergence results on Projection methods to the considered *Algorithm-1*.

Let now consider *Algorithm-1* for the fixed value of the index i . The orthogonal projector $P_{\mathcal{H}_i}$ onto \mathcal{H}_i defined by (19) has the form

$$(26) \quad P_{\mathcal{H}_i} = \frac{\mathbf{v}_i^{(k)}(\mathbf{v}_i^{(k)})^*}{\|\mathbf{v}_i^{(k)}\|_2^2}$$

and the oblique projector $Q_{\mathcal{H}_i}^{\mathcal{L}_i}$ onto \mathcal{H}_i along the orthogonal complement of \mathcal{L}_i defined by (20) has the form

$$(27) \quad Q_{\mathcal{H}_i}^{\mathcal{L}_i} = \mathbf{v}_i^{(k)}(\mathbf{w}_i^{(k)})^*.$$

If we consider the exact eigenpair $(\lambda_i, \mathbf{x}_i)$ as an approximate eigenpair of the operator A_m , then the following estimates hold for the corresponding residual vector. For the sake of simplicity, we omit the superscript (k) .

To investigate the distance between the Ritz value and the exact eigenvalue we can use the following theorem.

Theorem 1. *Let A be diagonalizable with eigenvalue decomposition (4)*

$$A = V\Lambda V^{-1},$$

where $\Lambda = \text{diag}\{\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}\}$. Let $(\frac{1}{\hat{\lambda}_i}, \hat{\mathbf{x}}_i)$ be an approximate eigenpair of A computed by *Algorithm-1* with the corresponding residual vector

$$\mathbf{r}_i = \left(A - \frac{1}{\hat{\lambda}_i} I \right) \hat{\mathbf{x}}_i, \quad (i = 1, \dots, n).$$

Then there exists an eigenvalue λ_i of A such that

$$(28) \quad \left| \frac{1}{\lambda_i} - \frac{1}{\hat{\lambda}_i} \right| \leq k(V) \|\mathbf{r}_i\|_2,$$

where $k(V) = \|V\|_2 \|V^{-1}\|_2$ denotes the condition number of V .

Proof: The result is obvious if $\lambda_i = \hat{\lambda}_i$. Let assume that $\lambda_i \neq \hat{\lambda}_i$, then the matrix $A - \frac{1}{\hat{\lambda}_i}I$ is nonsingular and we have

$$\left(A - \frac{1}{\hat{\lambda}_i}I\right)^{-1} = V \left(\Lambda - \frac{1}{\hat{\lambda}_i}I\right)^{-1} V^{-1},$$

which implies

$$1 \leq \|\hat{\mathbf{x}}_i\|_2 = \left\| \left(A - \frac{1}{\hat{\lambda}_i}I\right)^{-1} \mathbf{r}_i \right\|_2 = \left\| V \left(\Lambda - \frac{1}{\hat{\lambda}_i}I\right)^{-1} V^{-1} \mathbf{r}_i \right\|_2 \leq k(V) \frac{\|\mathbf{r}_i\|_2}{\min_j \left| \frac{1}{\lambda_j} - \frac{1}{\hat{\lambda}_i} \right|}.$$

Rearranging the terms we get the result of the theorem.

A similar approach to Theorem 1 for studying the convergence of Ritz values is the next theorem, where both left and right residuals are used. This is the two-sided analogous of the known Bauer-Fike theorem (see [1, 10]).

Theorem 2. Let we have the eigenvalue decomposition (4) of A

$$A = V \Lambda V^{-1},$$

where $\Lambda = \text{diag}\{\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}\}$. Let $(\frac{1}{\hat{\lambda}_i}, \hat{\mathbf{x}}_i, \hat{\mathbf{y}}_i)$ be an approximate eigentriplet of A computed by Algorithm-1 with the corresponding residual vectors

$$\mathbf{r}_i = \left(A - \frac{1}{\hat{\lambda}_i}I\right) \hat{\mathbf{x}}_i \text{ and } \mathbf{s}_i^* = \hat{\mathbf{y}}_i^* \left(A - \frac{1}{\hat{\lambda}_i}I\right), \quad (i = 1, \dots, n).$$

Then there exists an eigenvalue $\frac{1}{\lambda_i}$ of A such that

$$(29) \quad \left| \frac{1}{\lambda_i} - \frac{1}{\hat{\lambda}_i} \right| \leq \sqrt{k(V) \|\mathbf{r}_i\|_2 \|\mathbf{s}_i\|_2},$$

where $k(V)$ denotes the condition number of V .

Proof: If $\frac{1}{\lambda_i}$ is an eigenvalue of A the result is obvious. Let suppose the opposite. Then the matrix $A - \frac{1}{\hat{\lambda}_i}I$ is nonsingular and we have

$$1 = |\hat{\mathbf{y}}_i^* \hat{\mathbf{x}}_i| = \left| \mathbf{s}_i^* \left(A - \frac{1}{\hat{\lambda}_i}I\right)^{-2} \mathbf{r}_i \right| = \left| \mathbf{s}_i^* V \left(\Lambda - \frac{1}{\hat{\lambda}_i}I\right)^{-2} V^{-1} \mathbf{r}_i \right| \leq$$

by using that the matrix $(\Lambda - \frac{1}{\hat{\lambda}_i}I)$ is diagonal

$$\leq k(V) \|\mathbf{s}_i\|_2 \|\mathbf{r}_i\|_2 \left\| \left(\Lambda - \frac{1}{\hat{\lambda}_i}I\right)^{-2} \right\|_2,$$

which implies by rearranging the terms

$$\min_{\lambda_i} \left| \frac{1}{\lambda_i} - \frac{1}{\hat{\lambda}_i} \right| \leq \sqrt{k(V) \|\mathbf{s}_i\|_2 \|\mathbf{r}_i\|_2}.$$

From the last expression, we get the result of the theorem.

It is important to remark that the estimates obtained in Theorem 1 and Theorem 2 depend on the quantity $k(V)$ which is not known, and so these are *a priori* results. Such estimates are of great value in assessing the relative performance of algorithms.

6 Conclusion

Our goal in this study was to introduce a new iterative method for diagonalization of the inverse companion matrix. We have shown that this method can be considered as an oblique projection method. Using this we proved some analogs of known results from the theory of projection methods. We established an a posterior error bound and error bounds by using the residuals. We also show that the new iterative scheme is equivalent to the known Inverse Weierstrass iterative method for simultaneous approximations of all the zeros of a polynomial.

REFERENCES:

- [1] Y. Saad, *Numerical methods for large eigenvalue problems*, Classics in Applied Mathematics, vol. 66, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2011.
- [2] G. Stewart, *Matrix Algorithms, Volume II: Eigensystems*, SIAM, Philadelphia, 2001.
- [3] G.H. Golub, H.A. van der Vorst, Eigenvalue computation in the 20th century, *J.Comput.Appl.Math.* 123: 35-65, 2000.
- [4] A. Björck, G. Dahlquist, *Numerical Methods in Scientific Computing Volume II*, Society for Industrial and Applied Mathematics, Philadelphia, PA 2008.
- [5] B.N. Parlett, *The Symmetric Eigenvalue Problem*, Prentice-Hall, Englewood Cliffs, NJ, 1980.
- [6] J. Demmel, J. Dongarra, A. Ruhe, *Templates for the Solution of Algebraic Eigenvalue Problems: A Practical Guide*, Society fo Industrial and Applied Mathematics, PA, 2000.
- [7] G.L.G. Sleijpen, H.A. van der Vorst, A Jacobi-Davidson iteration method for linear eigenvalue problem, *SIAM J.Matrix Anal. Appl.*, 17: 401-425, 1996.
- [8] A. Ostrowski, On the convergence of the Rayleigh quotient iteration for the computation of the characteristic roots and vectors, I-VI, *Arch. Ration. Mech. Anal.* I 1:233-241, 1958; II 2:423-428, 1958; III 3:325-340, 1959; IV 3:341-347, 1959; V 3:472-481, 1959; VI 4:153-165, 1959.
- [9] B.N. Parlett, The Rayleigh quotient iteration and some generalizations for nonnormal matrices, *Math. Comput.* 28: 679-693, 1974.
- [10] I.N.Zwaan, M.E.Hochstenbach, Krylov-Schur-Type restarts for the two-sided Arnoldi method, *Siam J. Matrix Anal. Appl.*, Vol.38, No.2, pp.297-321, 2017.
- [11] G.H. Nedzhibov, Inverse Weierstrass-Durand-Kerner iterative method, *International Journal of Applied Mathematics*,ISSN:2051-5227 Vol. **28**, Issue 2, pp.1258-1264, (2013).