Annual of Konstantin Preslavsky University of Shumen
Faculty of Mathematics and Informatics , vol. XIX C, 2018, pp. 3- 20

ERROR ESTIMATES OF BEST PROXIMITY POINTS FOR REICH MAPS IN UNIFORMLY CONVEX BANACH SPACES*

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ABSTRACT: *We find a priori and a posteriori error estimates of the best proximity point, obtained with the Picard iteration associated to a cyclic Reich contraction map, which is defined on a uniformly convex Banach space with modulus of convexity of power type.*

KEYWORDS: *Coupled fixed points, Coupled best proximity points, Modular function space*

1 Introduction

In mathematical modeling of real world processes an approach for solving the arising problems is the Banach Contraction Principle, which is the fundamental result in fixed point theory. The theory of fixed points serves for solving of equations $Tx = x$ for mappings T defined on subsets of metric spaces or normed spaces. A generalization of the Banach Contraction Principle is the notion of cyclical maps [8], i.e. $T(A) \subseteq B$ and $T(B) \subseteq A$. Since a non-self mapping $T : A \to B$ does not necessarily have a fixed point, one often attempts to find an element x which is in some sense closest to Tx. Best proximity point theorems are relevant in this perspective. The notion of best proximity point is introduced in [4]. A sufficient condition for existence and the uniqueness of best proximity points in uniformly convex Banach spaces is given in [4]. Since publication [4] the problem for existence and uniqueness of best proximity point has been widely investigated and the research on this problem continues. Interesting examples of cyclic maps can be found in [7]. The results of [4] were generalized for modular function spaces

*The first author is partially supported by fund MU17-FMI-007 of University of Plovdiv Paisii Hilendarski.

in [14]. A connection between best proximity points and variational principles was found in [11].

There are many problems about fixed points and best proximity points that are not easy to be solved or could not be solved exactly. One of the advantages of Banach fixed point theorem is the error estimates of the successive iterations and the rate of convergence. That is why an estimation of the error when an iterative process is used is of interest. An extensive study about approximations of fixed points can be found in [2].

A first result in the approximation of the sequence of successive iterations, which converges to the best proximity point for cyclic contractions is obtained in [15]. The above mention results was expanded for coupled best proximity points [9], where ”a priori error estimates“ and ”a posteriori error estimates“ for the coupled best proximity points and for the coupled fixed points, which are obtained through a sequence of successive iterations were obtained.

We tried to generalize the results from [15] for cyclic Reich maps.

2 Preliminaries

In this section we give some basic definitions and concepts which are useful and related to the best proximity points. Let (X,ρ) be a metric space. Define a distance between two subset $A,B\subset X$ by $\mathrm{dist}(A,B)=\inf\{\rho(x,y):x\in A,y\in B\}$. Just to simplify the notations we will denote $\mathrm{dist}(A,B)$ with d.

Let A and B be nonempty subsets of a metric space (X,ρ). The map $T:A\bigcup B\to A\bigcup B$ is called a cyclic map if $T(A)\subseteq B$ and $T(B)\subseteq A$. A point $\xi\in A$ is called a best proximity point of the cyclic map T in A if $\rho(\xi,T\xi)=\mathrm{dist}(A,B)$.

Let A and B be nonempty subsets of a metric space (X,ρ). The map $T:A\bigcup B\to A\bigcup B$ is called a Kannan cyclic contraction map if T is a cyclic map and for some $k\in(0,1/2)$ there holds the inequality $\rho(Tx,Ty)\le k(\rho(Tx,x)+\rho(Ty,y))+(1-2k)d$ for any $x\in A$, $y\in B$. The definition for cyclic Kannan contraction is introduced and investigated in [13].

Following [6] we will define a generalization of cyclic Reich contraction maps.

Definition 1. *Let (X,ρ) be a metric space and $A,B \subset X$. A cyclic map $T: A \to B$ and $T: B \to A$ is called cyclic Reich contraction maps if there exist nonegative constants k_i, $i = 1,2,3,4,5$, satisfying $0 < \sum_{i=1}^{5} k_i < 1$ such that for each $x \in A$ and $y \in B$ the inequality*

$$\rho(Tx,Ty) \le k_1\rho(x,y) + k_2\rho(x,Tx) + k_3\rho(y,Ty) + k_4\rho(x,Ty) + \\ + k_5\rho(y,Tx) + \left(1 - \sum_{i=1}^{5} k_i\right) d$$

holds.

As pointed in [6] from the symmetry of the function ρ it follows that $k_2 = k_3$ and $k_4 = k_5$. Therefore if T is a cyclic Reich contraction maps then there exist $a_1, a_2, a_3 \ge 0$, such that $0 < a_1 + 2a_2 + 2a_3 < 1$ and there holds the inequality

$$\rho(Tx,Ty) \le a_1\rho(x,y) + a_2(\rho(x,Tx) + \rho(y,Ty)) + \\ + a_3(\rho(x,Ty) + \rho(y,Tx)) + (1-a))d,$$

where $a = a_1 + 2a_2 + 2a_3$.

The best proximity results need norm-structure of the space X. When we investigate a Banach space $(X, \|\cdot\|)$ we will always consider the distance between the elements to be generated by the norm $\|\cdot\|$ i.e. $\rho(x,y) = \|x-y\|$. We will denote the unit sphere and the unit ball of a Banach space $(X, \|\cdot\|)$ by S_X and B_X respectively.

The assumption that the Banach space $(X, \|\cdot\|)$ is uniformly convex plays a crucial role in the investigation of best proximity points [4].

Definition 2. *Let $(X, \|\cdot\|)$ be a Banach space. For every $\varepsilon \in (0,2]$ we define the modulus of convexity of $\|\cdot\|$ by*

$$\delta_{\|\cdot\|}(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x-y\| \geq \varepsilon\right\}.$$

The norm is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. The space $(X, \|\cdot\|)$ is then called uniformly convex space.

The main result from [13] is the next theorem.

Theorem 3. *([13]) Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space and $T : A \cup B \to A \cup B$ be a cyclic Kannan contraction map. Then there is a unique best proximity point ξ of T in A, $T\xi$ is a unique best proximity point of T in B and $\xi = T^2\xi = T^{2n}\xi$. Further if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}_{n=1}^{\infty}$ converges to ξ and $\{x_{2n+1}\}_{n=0}^{\infty}$ converges to $T\xi$.*

We will use the following two lemmas, established in [4], for proving the uniqueness of the best proximity points.

Lemma 4. ([4]) Let A be a nonempty, closed, convex subset, and B be a nonempty, closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in A and $\{y_n\}_{n=1}^{\infty}$ be a sequence in B satisfying:
1) $\lim\limits_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A,B)$;
2) for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$, such that for all $m > n \geq N_0$, $\|x_n - y_n\| \leq \operatorname{dist}(A,B) + \varepsilon$.

Then for every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$, such that for all $m > n > N_1$, holds $\|x_m - z_n\| \leq \varepsilon$.

Lemma 5. ([4]) Let A be a nonempty, closed, convex subset, and B be a nonempty, closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in A and $\{y_n\}_{n=1}^{\infty}$ be a sequence in B satisfying: $\lim\limits_{n\to\infty} \|x_n - y_n\| = \operatorname{dist}(A,B)$ and $\lim\limits_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A,B)$, then $\lim\limits_{n\to\infty} \|x_n - z_n\| = 0$.

The idea of best proximity points was further generalized in modular function spaces for cyclic contractions in [14] and for cyclic Kannan contraction maps in [10]

We will need the following technical lemma:

Lemma 6. *([13]) Let $(X,\|\cdot\|)$ be a uniformly convex Banach space, A and B be nonempty closed and convex subsets of X. Let $T: A\cup B \to A\cup B$ be a cyclic map satisfying*

$$\|Tx-T^2x\|-d \le \alpha\left(\|x-Tx\|-d\right)$$

for all $x\in A\cup B$, for some $\alpha\in[0,1)$. Then

(i) $\|T^nx-T^{n+1}\|-d \le \alpha^n\left(\|x-Tx\|-d\right)$ *for all $x\in A\cup B$;*

(ii) $\lim_{n\to\infty}\|T^nx-T^{n+1}x\| = d$ *for all $x\in A\cup B$;*

(iii) $\lim_{n\to\infty}\|T^{2n}x-T^{2n\pm2}x\| = 0$ *for all $x\in A\cup B$;*

(iv) z is a best proximity point of T if and only if z is a fixed point for T^2.

Let us point out that the inequality in (i) holds not only for uniformly convex Banach spaces but also for any metric space.

For any uniformly convex Banach space X there holds the inequality

$$\left\|\frac{x+y}{2}-z\right\| \le \left(1-\delta_X\left(\frac{r}{R}\right)\right)R \tag{1}$$

for any $x,y,z\in X$, $R>0$, $r\in[0,2R]$, $\|x-z\|\le R$, $\|y-z\|\le R$ and $\|x-y\|\ge r$.

If $(X,\|\cdot\|)$ is a uniformly convex Banach space, then $\delta_X(\varepsilon)$ is strictly increasing function. Therefore if $(X,\|\cdot\|)$ is a uniformly convex Banach space then there exists the inverse function δ^{-1} of the modulus of convexity. If there exist constants $C>0$ and $q>0$, such that the inequality $\delta_{\|\cdot\|}(\varepsilon)\geq C\varepsilon^q$ holds for every $\varepsilon\in(0,2]$ we say that the modulus of convexity is of power type q. It is well known that for any Banach space and for any norm there holds the inequality $\delta(\varepsilon)\leq K\varepsilon^2$. The modulus of convexity with respect to the canonical norm $\|\cdot\|_p$ in ℓ_p or L_p is $\delta_{\|\cdot\|_p}(\varepsilon)=1-\sqrt[p]{1-\left(\frac{\varepsilon}{2}\right)^p}$ for $p\geq 2$ and the modulus of convexity $\delta_{\|\cdot\|_p}(\varepsilon)$ is the solution of the equation $\left(1-\delta+\frac{\varepsilon}{2}\right)^p+\left|1-\delta-\frac{\varepsilon}{2}\right|^p=2$ for $1<p<2$. It is well known that the modulus of convexity with respect to the canonical norm in ℓ_p or L_p is of power type and there holds the inequalities $\delta_{\|\cdot\|_p}(\varepsilon)\geq\frac{\varepsilon^p}{p2^p}$ for $p\geq 2$ and $\delta_{\|\cdot\|_p}(\varepsilon)\geq\frac{(p-1)\varepsilon^2}{8}$ for $p\in(1,2)$ [12].

An extensive study of the Geometry of Banach spaces can be found in [1, 3, 5].

3 Error Estiamtes for Best Proximity Points

We will need the following lemma.

Lemma 7. *Let A and B be nonempty subsets of a metric space (X,ρ) and let $T:A\cup B\to A\cup B$ be a cyclic Reich contraction map. Then for every $x\in A\cup B$ there holds the inequalities $\rho(T^nx,T^{n+1}x)-d\leq\alpha^n\left(\rho(x,Tx)-d\right)$ and $\rho(T^nx,T^{n+1}x)-d\leq\alpha\left(\rho(T^{n-1}x,T^nx)-d\right)$, where $\alpha=\frac{a_1+a_2+a_3}{1-a_2-a_3}\in[0,1)$.*

Proof. From the inequality

$$\begin{aligned}\rho(T^nx,T^{n+1}x) &\leq a_1\rho(T^{n-1}x,T^nx)+a_2(\rho(T^nx,T^{n-1}x)+\\ &\quad+\rho(T^{n+1}x,T^nx))+a_3\rho(T^{n+1}x,T^{n-1}x)+\\ &\quad+(1-a_1-2(a_2+a_3))d\end{aligned}$$

we get the inequality

$$\begin{aligned}(1-a_2)\rho(T^nx,T^{n+1}x) &\leq (a_1+a_2)(\rho(T^{n-1}x,T^nx)+\\ &\quad+a_3(\rho(T^{n+1}x,T^nx)+\rho(T^nx,T^{n-1}x)+\\ &\quad+(1-a_1-2(a_2-a_3))d.\end{aligned}$$

Fro the above inequality we get

$$(1-a_2-a_3)\rho(T^n x, T^{n+1}x) \leq (a_1+a_2+a_3)(\rho(T^{n-1}x, T^n x)+ \\ +(1-a_1-2(a_2-a_3))d,$$

$$\rho(T^n x, T^{n+1}x) \leq \frac{a_1+a_2+a_3}{1-a_2-a_3}\rho(T^{n-1}x, T^n x) + \frac{1-a_1-2(a_2-a_3)}{1-a_2-a_3}d$$

and

$$\rho(T^n x, T^{n+1}x) \leq \frac{a_1+a_2+a_3}{1-a_2-a_3}\rho(T^{n-1}x, T^n x) + \left(1-\frac{a_1+a_2+a_3}{1-a_2-a_3}\right)d.$$

Let us put $\alpha = \frac{a_1+a_2+a_3}{1-a_2-a_3}$. It is easy to check that $\alpha \in [0,1)$ for every $a_1, a_2, a_3 \in [0,1)$, such that $a_1+2(a_2+a_3) \in [0,1)$. Consequently there holds

$$\rho(T^n x, T^{n+1}x) \leq \alpha\rho(T^{n-1}x, T^n x) + (1-\alpha)d$$

and therefore

$$\rho(T^n x, T^{n+1}x) - d \leq \alpha\left(\rho(T^{n-1}x, T^n x) - d\right).$$

Applying Lemma 6 we get

$$\rho(T^n x, T^{n+1}x) - d \leq \alpha^n\left(\rho(x, Tx) - d\right).$$

□

We will use in the sequel $\alpha = \frac{a_1+a_2+a_3}{1-a_2-a_3}$.

It seems that the technique applied in [13] for cyclic Kannan contraction maps can be applied and for a wide classes of cyclic Reich maps namely cyclic maps that satisfy the inequality

(2)
$$\begin{aligned} \|Tx-Ty\| \quad \leq \quad & a_1\|x-y\| + a_2\left(\|x-Tx\| + \|y-Ty\|\right) \\ & +a_3\left(\|x-Ty\| + \|y-Tx\|\right) + (1-a_1-2a_2-4a_3))d \end{aligned}$$

for any $x \in A$ and $y \in B$.

Let us point out that any cyclic Reich map with $a_3 = 0$ satisfies (2).

Lemma 8. *Let A and B be nonempty subsets of a metric space* (X,ρ) *and let* $T : A\cup B \to A\cup B$ *be a cyclic map, that satisfies (2). Then for every* $x\in A\cup B$ *there holds the inequalities* $\rho(T^n x, T^{n+1}x) - d \le \alpha^n(\rho(x,Tx)-d)$ *and* $\rho(T^n x, T^{n+1}x) - d \le \alpha\big(\rho(T^{n-1}x, T^n x)-d\big)$.

Proof. From the inequality

$$\begin{aligned}\rho(T^n x, T^{n+1}x) &\le a_1\rho(T^{n-1}x,T^n x)+a_2(\rho(T^n x,T^{n-1}x)+\\ &\quad +\rho(T^{n+1}x,T^n x))+a_3\rho(T^{n+1}x,T^{n-1}x)+\\ &\quad +(1-a-2a_3)d\le\\ &\le a_1\rho(T^{n-1}x,T^n x)+a_2(\rho(T^n x,T^{n-1}x)+\\ &\quad +\rho(T^{n+1}x,T^n x))+a_3\rho(T^{n+1}x,T^{n-1}x)+(1-a)d\end{aligned}$$

it follows that T satisfies the condition of Lemma 7. □

Lemma 9. *Let* $(X,\|\cdot\|)$ *be a uniformly convex Banach space, A and B be nonempty subsets of X and* $T : A\cup B\to A\cup B$ *be a cyclic Reich contraction map. Let* $x\in A$ *be such that there is a subsequence* $\{T^{2n_i}x\}_{i=1}^{\infty}$ *of* $\{T^{2n}x\}_{n=1}^{\infty}$*, which converges to* $z\in A$*. Then z is a best proximity point of T in A.*

Proof. Let $x\in A$. Let holds $\lim_{i\to\infty}T^{2n_i}x = z$ for some subsequence $\{T^{2n_i}x\}_{i=1}^{\infty}$ of $\{T^{2n}x\}_{n=1}^{\infty}$. From Lemma 7 it follows that we can apply

Lemma 6 and thus $\lim_{i\to\infty} \|T^{2n_i}x - T^{2n_i+1}x\| = d$. From the inequality

$$
\begin{aligned}
\|z - Tz\| &= \lim_{n\to\infty} \|T^{2n_i}x - Tz\| \\
&\leq \lim_{n\to\infty} \left(a_1\|T^{2n_i-1}x - z\| + a_2\left(\|T^{2n_i}x - T^{2n_i-1}x\| + \|Tz - z\|\right)\right) \\
&+ \lim_{n\to\infty} \left(a_3\left(\|T^{2n_i}x - z\| + \|T^{2n_i-1}x - Tz\|\right)\right) + (1-a)d \\
&\leq \lim_{n\to\infty} \left(a_1\|T^{2n_i-1}x - T^{2n_i}x\| + a_2\left(\|T^{2n_i}x - T^{2n_i-1}x\| + \|Tz - z\|\right)\right) \\
&+ \lim_{n\to\infty} \left(a_3\left(\|T^{2n_i}x - z\| + \|T^{2n_i-1}x - Tz\|\right)\right) + (1-a)d \\
&= a_1 d + a_2\left(d + \|Tz - z\|\right) + a_3 \lim_{n\to\infty} \|T^{2n_i-1}x - Tz\| + (1-a)d \\
&= a_2\|Tz - z\| + a_3 \lim_{n\to\infty} \|T^{2n_i-1}x - Tz\| + (1 - a_2 - 2a_3)d \\
&\leq a_2\|Tz - z\| + a_3 \lim_{n\to\infty} \left(\|T^{2n_i-1}x - z\| + \|z - Tz\|\right) + \\
&(1 - a_2 - 2a_3)d \\
&= a_2\|Tz - z\| + a_3 \lim_{n\to\infty} \left(\|T^{2n_i-1}x - T^{2n_i}x\| + \|z - Tz\|\right) + \\
&(1 - a_2 - 2a_3)d = (a_2 + a_3)\|Tz - z\| + (1 - a_2 - a_3)d
\end{aligned}
$$

we get that $\|z - Tz\| \leq d$, therefore $\|z - Tz\| = d$, i.e. z is a best proximity point of T in A. □

Lemma 10. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, A and B be nonempty subsets of X and $T : A \cup B \to A \cup B$ be a cyclic Reich contraction map, that satisfies (2). Let $x \in A$ be such that there is a subsequence $\{T^{2n_i}x\}_{i=1}^{\infty}$ of $\{T^{2n}x\}_{n=1}^{\infty}$, which converges to $z \in A$. Then z is the unique best proximity point of T in A.*

Proof. As far as any cyclic Reich map that satisfies (2) is a cylcic Reich map the proof that z is a best proximity point of T in A follows from Lemma 9.

Let us suppose that there is another best proximity point $w \neq z$ of T in A.

Case I) Let first $a_3 = 0$. By Lemma 6 it follows that $T^2w = w$ and we get the inequality

$$\begin{array}{rcl}\|w - Tz\| & = & \|T^2w - Tz\| \\ & \leq & a_1\|Tw - z\| + a_2(\|T^2w - Tw\| + \|Tz - z\|) + \\ & & (1 - a_1 - 2a_2)d \\ & \leq & a_1\|Tw - z\| + 2a_2d + (1 - a_1 - 2a_2)d = a_1\|Tw - z\| + \\ & & (1 - a_1)d.\end{array}$$

By similar arguments we get

$$\begin{array}{rcl}\|Tw - z\| & = & \|Tw - T^2z\| \\ & \leq & a_1\|w - Tz\| + a_2(\|T^2z - Tz\| + \|Tw - w\|) + \\ & & (1 - a_1 - 2a_2)d \\ & \leq & a_1\|w - Tz\| + 2a_2d + (1 - a_1 - 2a_2)d = a_1\|w - Tz\| + \\ & & (1 - a_1)d.\end{array}$$

After summing the last two inequalities we get

$$(1 - a_1)(\|Tw - z\| + \|w - Tz\|) \leq 2(1 - a_1)d$$

and therefore $\|Tw - z\| = \|w - Tz\| = d$. Applying Lemma 5 we get that $w = z$.

Case II) Let now T satisfies (2). From the assumption that T is a cyclic Reich contraction it follows from Lemma 7 that we can apply Lemma 6 and thus $T^2w = w$. We get the inequality

$$\begin{array}{rcl}\|w - Tz\| & = & \|T^2w - Tz\| \\ & \leq & a_1\|Tw - z\| + a_2(\|T^2w - Tw\| + \|Tz - z\|) + \\ & & a_3(\|T^2w - z\| + \|Tz - Tw\|) + (1 - a - 2a_3)d \\ & \leq & a_1\|Tw - z\| + a_2(\|w - Tw\| + \|Tz - z\|) \\ & & +a_3(\|T^2w - Tw\| + \|Tw - z\| + \|Tz - w\| + \|w - Tw\|) \\ & & +(1 - a - 2a_3)d \\ & \leq & a_1\|Tw - z\| + a_2(d + d) + \\ & & a_3(d + \|Tw - z\| + \|Tz - w\| + d) + (1 - a - 2a_3)d \\ & = & (a_1 + a_3)\|Tw - z\| + a_3\|Tz - w\| + (1 - a_1 - 2a_3)d.\end{array}$$

By similar arguments we get

$$\begin{aligned}
\|Tw-z\| &= \|Tw-T^2z\| \\
&\leq a_1\|w-Tz\|+a_2(\|T^2z-Tz\|+\|Tw-w\|)+ \\
&\quad a_3(\|T^2z-w\|+\|Tw-Tz\|)+(1-a-2a_3)d \\
&\leq a_1\|w-Tz\|+a_2(\|z-Tz\|+\|Tw-w\|) \\
&\quad +a_3(\|T^2z-Tz\|+\|Tz-w\|+\|Tw-z\|+\|z-Tz\|)+ \\
&\quad (1-a-2a_3)d \\
&\leq a_1\|w-Tz\|+a_2(d+d)+ \\
&\quad a_3(d+\|Tz-w\|+\|Tw-z\|+d)+(1-a-2a_3)d \\
&= (a_1+a_3)\|w-Tz\|+a_3\|Tw-z\|+(1-a_1-2a_3)d.
\end{aligned}$$

After summing the leat two inequalities we get

$$(1-a_1-2a_3)\left(\|Tw-z\|+\|w-Tz\|\right)\leq 2(1-a_1-a_3)d$$

and therefore $\|Tw-z\|=\|w-Tz\|=d$. Applying Lemma 5 we get that $w=z$. □

Theorem 11. *Let A and B be nonempty, closed and convex subsets of a uniformly convex Banach $(X,\|\cdot\|)$ space, such that $d=\operatorname{dist}(A,B)>0$, and let there exist $C>0$ and $q\geq 2$, such that $\delta_{\|\cdot\|}(\varepsilon)\geq C\varepsilon^q$. Let $T: A\cup B\to A\cup B$ be a cyclic Reich contraction map that satisfies (2). Then*

1. *there exists a unique best proximity point ξ of T in A, $T\xi$ is unique best proximity point of T in B and $\xi=T^2\xi=T^{2n}\xi$;*
2. *for any $x_0\in A$ the sequence $\{x_{2n}\}_{n=1}^{\infty}$ converges to ξ and $\{x_{2n+1}\}_{n=1}^{\infty}$ converges to $T\xi$, where $x_{n+1}=Tx_n$, $n=0,1,2,\ldots$;*
3. *a priori error estimate holds*

$$\|\xi-T^{2n}x\|\leq\frac{\|x-Tx\|}{1-\sqrt[q]{\alpha^2}}\sqrt[q]{\frac{\|x-Tx\|-d}{Cd}}\left(\sqrt[q]{\alpha}\right)^{2n}; \qquad (3)$$

4. *a posteriori error estimate holds*
(4)
$$\|T^{2n}x-\xi\| \leq \frac{\|T^{2n-1}x-T^{2n}x\|}{1-\sqrt[q]{\alpha^2}}\sqrt[q]{\frac{\|T^{2n-1}x-T^{2n}x\|-d}{Cd}}\sqrt[q]{\alpha}.$$

Proof. (i) By Lemma 6 it follows that $\lim_{n\to\infty}\|T^{2n}x-T^{2n+1}x\| = d$. Let us suppose that there is $\varepsilon_0 > 0$ so that for every $k \in \mathbb{N}$ there are $m_k > n_k \geq k$ such that

$$\|T^{2m_k}x-T^{2n_k+1}x\| > d+\varepsilon_0.$$

Let m_k be the smallest integer greater than n_k to satisfy the above inequality, i.e.

$$\|T^{2(m_k-1)}x-T^{2n_k+1}x\| \leq d+\varepsilon_0.$$

By the inequality

$$d+\varepsilon_0 < \|T^{2m_k}x-T^{2n_k+1}x\| \leq \|T^{2m_k}x-T^{2m_k-2}x\|+\|T^{2m_k-2}x-T^{2n_k+1}x\|$$

and Lemma 6 after taking a limit on $k\to\infty$ we get

$$d+\varepsilon_0 \leq \|T^{2m_k}x-T^{2n_k+1}x\| \leq d+\varepsilon_0,$$

i.e. $\lim_{k\to\infty}\|T^{2m_k}x-T^{2n_k+1}x\| = d+\varepsilon_0$.

On the other hand by

$$\|T^{2m_k}x-T^{2n_k+1}x\| \leq \|T^{2m_k}x-T^{2m_k+2}x\|+\|T^{2m_k+2}x-T^{2n_k+3}x\|+ \|T^{2n_k+3}x-T^{2n_k+1}x\|$$

and Lemma 6 after taking a limit on $k\to\infty$ we get

$$d+\varepsilon_0 \leq \lim_{k\to\infty}\|T^{2m_k}x-T^{2n_k+1}x\| \leq \lim_{k\to\infty}\|T^{2m_k+2}x-T^{2n_k+3}x\|.$$

From Lemma 6 we have that

$$\lim_{k\to\infty}\|T^{2m_k+1}x-T^{2n_k+2}x\| \leq \alpha\lim_{k\to\infty}\|T^{2m_k}x-T^{2n_k+1}x\|+(1-\alpha)d \leq d+\varepsilon_0.$$

using the last inequality we get the chain of inequalities

$$\begin{aligned}
d+\varepsilon_0 &\leq \lim_{k\to\infty}\|T^{2m_k+2}x-T^{2n_k+3}x\|\\
&\leq \lim_{k\to\infty}\big(a_1\|T^{2m_k+1}x-T^{2n_k+2}x\|+a_2\big(\|T^{2m_k+2}x-T^{2m_k+1}x\|+\\
&\quad \|T^{2n_k+3}x-T^{2n_k+2}x\|\big)\big)+a_3\lim_{k\to\infty}\big(\|T^{2m_k+2}x-T^{2n_k+2}x\|+\\
&\quad \|T^{2n_k+3}x-T^{2m_k+1}x\|\big)+(1-a)d\\
&\leq a_1(d+\varepsilon_0)+2a_2d+a_3\lim_{k\to\infty}\big(\|T^{2m_k+2}x-T^{2m_k+1}x\|+\\
&\quad \|T^{2m_k+1}x-T^{2n_k+2}x\|\big)+a_3\lim_{k\to\infty}\big(\|T^{2n_k+3}x-T^{2n_k+2}x\|+\\
&\quad \|T^{2n_k+2}x-T^{2m_k+1}x\|\big)+(1-a)d\\
&= (a_1+2a_3)(d+\varepsilon_0)+2(a_2+a_3)d+(1-a)d
\end{aligned}$$

and thus we get

$$(1-a_1-2a_3)(d+\varepsilon_0)\leq(1-a_1-2a_3)d,$$

which is a contradiction and therefore for every $\varepsilon>0$ there is $N\in\mathbb{N}$ so that the inequality $\|T^{2m}x-T^{2n+1}x\|<d+\varepsilon$ holds for any $m>n\geq N$. From Lemma 4 it follows that $\{x_{2n}\}_{n=1}^{\infty}$ is a Cauchy sequence and hence convergent. Consequently there is $\xi\in A$ so that $\lim_{n\to\infty}x_{2n}=\xi$. By Lemma 10 it follows that ξ is the unique best proximity point of T in A. From Lamma 6 it follows that ξ is a fixed point for T^2.

It remains to prove that $T\xi$ is a best proximity point of T in B. From Lemma 6 we have the inequities

$$d\leq\|T^2\xi-T\xi\|\leq\alpha\|T\xi-\xi\|+(1-\alpha)d=d$$

and therefore $T\xi$ is a best proximity point of T in B.

The proof that $T\xi$ is a unique best proximity point can be done similarly to the proof that ξ is a unique best proximity point.

(ii) For any $x\in A$ we have that the sequence $\{T^{2n}x\}_{n=1}^{\infty}$ converges to a best proximity point of T in A. By the uniqueness of the best proximity point of T in A it follows that $\{T^{2n}x\}_{n=1}^{\infty}$ converges to ξ.

From the inequalities

$$\begin{aligned}\|T^{2n+1}x-\xi\| &= \|T^{2n+1}x-T^2\xi\| \le a_1\|T^{2n}x-\xi\|+ \\ & a_2\left(\|T^{2n+1}x-T^{2n}x\|+\|T\xi-\xi\|\right)+ \\ & a_3\left(\|T^{2n+1}x-\xi\|+\|T\xi-T^{2n}x\|\right)+(1-a-2a_3)d \\ &\le a_1\|T^{2n}x-\xi\|+a_2\left(\|T^{2n+1}x-T^{2n}x\|+\|T\xi-\xi\|\right) \\ & +a_3\left(\|T^{2n+1}x-\xi\|+\|T\xi-T^{2n}x\|\right)+(1-a)d\end{aligned}$$

we get that $(1-a_3)\|T^{2n+1}x-T^2\xi\| \le (1-a_3)d$. Using the equality $\|T\xi-T^2\xi\|=d$ and Lemma 5 it follows that $\{T^{2n+1}x\}_{n=1}^{\infty}$ converges to $T\xi$.

(iii) We will prove first that for any $x \in A$, $n \in \mathbb{N}$ and $l \le 2n$ there holds the inequality

$$\delta_{\|\cdot\|}\left(\frac{\|T^{2n}x-T^{2n+2}x\|}{d+k^l(\|T^{2n-l}x-T^{2n+1-l}x\|-d)}\right) \le \frac{k^l(\|T^{2n-l}x-T^{2n+1-l}x\|-d)}{d+k^l(\|T^{2n-l}x-T^{2n+1-l}x\|-d)}.$$

Indeed let $x \in A$ be arbitrary chosen. From Lemma 7 we have the inequalities

$$\|T^{2n}x-T^{2n+1}x\| \le d+\alpha^l(\|T^{2n-l}x-T^{2n+1-l}x\|-d),$$

$$\|T^{2n+2}x-T^{2n+1}x\| \le d+\alpha^{l+1}(\|T^{2n-l}x-T^{2n+1-l}x\|-d) < d+ \alpha^l(\|T^{2n-l}x-T^{2n+1-l}x\|-d)$$

and

$$\begin{aligned}\|T^{2n+2}x-T^{2n}x\| &\le \|T^{2n+2}x-T^{2n+1}x\|+\|T^{2n+1}x-T^{2n}x\| \\ &\le 2\left(d+\alpha^l(\|T^{2n-l}x-T^{2n+1-l}x\|-d)\right).\end{aligned}$$

After a substitution in (1) with $x=T^{2n}x$, $y=T^{2n+2}x$, $z=T^{2n+1}x$, $r=\|T^{2n+2}x-T^{2n}x\|$ and $R=d+\alpha^l\left(\|T^{2n-l}x-T^{2n+1-l}x\|-d\right)$ and using the convexity of the set A we get the chain of inequalities

$$(5)\quad \begin{aligned} d &\le \left\| \frac{T^{2n}x+T^{2n+2}x}{2} - T^{2n+1}x \right\| \\ &\le \left(1-\delta_{\|\cdot\|}\left(\frac{\|T^{2n}x-T^{2n+2}x\|}{d+\alpha^l(\|T^{2n-l}x-T^{2n+1-l}x\|-d)}\right)\right) \\ &\quad \left(d+\alpha^l(\|T^{2n-l}x-T^{2n+1-l}x\|-d)\right). \end{aligned}$$

From (5) we obtain the inequality

$$(6)\quad \begin{aligned} &\delta_{\|\cdot\|}\left(\frac{\|T^{2n}x-T^{2n+2}x\|}{d+\alpha^l(\|T^{2n-l}x-T^{2n+1-l}x\|-d)}\right) \le \\ &\frac{\alpha^l(\|T^{2n-l}x-T^{2n+1-l}x\|-d)}{d+\alpha^l(\|T^{2n-l}x-T^{2n+1-l}x\|-d)}. \end{aligned}$$

From the uniform convexity of X is follows that $\delta_{\|\cdot\|}$ is strictly increasing and therefore there exists its inverse function $\delta_{\|\cdot\|}^{-1}$, which is strictly increasing too.

Just to fit some of the next formulas in the text field let us put $S_{n,m}(x) = \|T^n x - T^m x\| - d$. From (6) we get

$$(7)\quad \|T^{2n}x-T^{2n+2}x\| \le \left(d+\alpha^l S_{2n-l,2n+1-l}\right)\delta_{\|\cdot\|}^{-1}\left(\frac{\alpha^l S_{2n-l,2n+1-l}}{d+\alpha^l S_{2n-l,2n+1-l}}\right).$$

By the inequality $\delta_{\|\cdot\|}(t) \ge Ct^q$ it follows that $\delta_{\|\cdot\|}^{-1}(t) \le \left(\frac{t}{C}\right)^{1/q}$. From (7) and the inequalities

$$d \le d+\alpha^l(\|T^{2n-l}x-T^{2n+1-l}x\|-d) \le \|T^{2n-l}x-T^{2n+1-l}x\|$$

we obtain

$$(8)\quad \begin{aligned} \|T^{2n}x-T^{2n+2}x\| &\le \left(d+\alpha^l S_{2n-l,2n+1-l}\right)\sqrt[q]{\frac{\alpha^l S_{2n-l,2n+1-l}}{C.(d+\alpha^l S_{2n-l,2n+1-l})}} \\ &\le \|T^{2n-l}x-T^{2n+1-l}x\|\sqrt[q]{\frac{S_{2n-l,2n+1-l}}{Cd}}\left(\sqrt[q]{\alpha}\right)^l. \end{aligned}$$

From (i) and (ii) there exists a unique ξ, such that $\|\xi - T\xi\| = d$, $T^2\xi = \xi$ and ξ is a limit of the sequence $\{T^{2n}x\}_{n=1}^{\infty}$ for any $x \in A$.

After a substitution with $l = 2n$ in (8) we get the inequality

$$\sum_{n=1}^{\infty} \left\|T^{2n}x - T^{2n+2}x\right\| \le \|x - Tx\| \sqrt[q]{\frac{\|x - Tx\| - d}{Cd}} \sum_{n=1}^{\infty} \left(\sqrt[q]{\alpha}\right)^{2n}$$
$$= \|x - Tx\| \sqrt[q]{\frac{\|x - Tx\| - d}{Cd}} \cdot \frac{\sqrt[q]{\alpha^2}}{1 - \sqrt[q]{\alpha^2}}$$

and consequently the series $\sum_{n=1}^{\infty}(T^{2n}x - T^{2n+2}x)$ is absolutely convergent. Thus for any $m \in \mathbb{N}$ there holds $\xi = T^{2m}x - \sum_{n=m}^{\infty}\left(T^{2n}x - T^{2n+2}x\right)$ and therefore we get the inequality

$$\left\|\xi - T^{2m}x\right\| \le \sum_{n=m}^{\infty} \left\|T^{2n}x - T^{2n+2}x\right\| \le$$
$$\|x - Tx\| \sqrt[q]{\frac{\|x - Tx\| - d}{Cd}} \cdot \frac{\left(\sqrt[q]{\alpha}\right)^{2m}}{1 - \sqrt[q]{\alpha^2}}.$$

(iv) After a substitution with $l = 1 + 2i$ in (8) we obtain

$$\|T^{2n+2i}x - T^{2n+2(i+1)}x\| \le$$
$$\|T^{2n-1}x - T^{2n}x\| \sqrt[q]{\frac{\|T^{2n-1}x - T^{2n}x\| - d}{Cd}} \left(\sqrt[q]{\alpha}\right)^{1+2i}. \tag{9}$$

From (9) we get that there holds the inequality

(10)

$$\|T^{2n}x - T^{2(n+m)}x\| \le \sum_{i=0}^{m-1} \|T^{2n+2i}x - T^{2n+2(i+1)}x\|$$
$$\le \sum_{i=0}^{m-1} \|T^{2n-1}x - T^{2n}x\| \sqrt[q]{\frac{\|T^{2n-1}x - T^{2n}x\| - d}{Cd}} \left(\sqrt[q]{\alpha}\right)^{1+2i}$$
$$= \|T^{2n-1}x - T^{2n}x\| \sqrt[q]{\frac{\|T^{2n-1}x - T^{2n}x\| - d}{Cd}} \sum_{i=0}^{m-1} \left(\sqrt[q]{\alpha}\right)^{1+2i}$$
$$= \|T^{2n-1}x - T^{2n}x\| \sqrt[q]{\frac{\|T^{2n-1}x - T^{2n}x\| - d}{Cd}} \cdot \frac{1 - \left(\sqrt[q]{\alpha}\right)^{2m}}{1 - \sqrt[q]{\alpha^2}} \sqrt[q]{\alpha}$$

and after letting $m \to \infty$ in (10) we obtain the inequality

$$\left\|T^{2n}x - \xi\right\| \leq \|T^{2n-1}x - T^{2n}x\| \sqrt[q]{\frac{\|T^{2n-1}x - T^{2n}x\| - d}{Cd}} \frac{\sqrt[q]{\alpha}}{1 - \sqrt[q]{\alpha^2}}.$$

□

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