COUPLED FIXED POINTS RESULTS FOR HARDY–ROGERS TYPE OF MAPS WITH THE MIXED MONOTONE PROPERTY OBTAINED WITH THE HELP OF A VARIATIONAL TECHNIQUE

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ABSTRACT: *Sufficient conditions have been obtained for the existence and uniqueness of coupled fixed points for Hardy-Rodgers maps with the mixed monotone property in partially ordered metric spaces using a variational technique. The obtained result generalizes and enriches already known results for other types of maps with mixed monotone property. It is shown that in partially ordered metric spaces the maps of Hardy-Rodgers and Reich with mixed monotonic properties do not coincide.*

KEYWORDS: *Coupled fixed points, Mixed monotone property, Variational principle*

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1 Introduction

Fixed point theorems, initiated by Banach's Contraction Principle [1] has proved to be a powerful tool in nonlinear analysis. We can not mention all kinds of generalizations of Banach's Contraction Principle. One direction for generalization of it is the notion of coupled fixed points [13], where mixed monotone maps in partially ordered by a cone Banach spaces are investigated. Later this idea was developed for mixed monotone maps in partially ordered metric spaces [3]. It is impossible to summarize all generalizations of the ideas of coupled fixed points, for mixed monotone maps, in partially ordered metric spaces. The investigation on the subject continuous as seen [2, 15, 21], which is far from exhausting the most recent results. Another kind of maps considered in partially ordered complete metric spaces are for monotone maps without the mixed monotone property [7, 8, 17, 22].

Another direction is by altering the contraction map conditions. Some classical type of contractive type conditions are Kannan maps [18], Chaterjea maps [5], Zamfirescu maps [24], Reich [23], Hardy–Rogers [14], ets.

Reich investigated in [23] maps that satisfy the contractive type condition

$$\rho(Tx,Ty)\le k_1\rho(x,y)+k_2\rho(Tx,x)+k_3\rho(Ty,x)+k_4\rho(Tx,y)+k_5\rho(Ty,x), \tag{1}$$

where $\sum_{i=1}^{5}k_i\in[0,1)$ and $k_i\ge 0$ for $i=1,2,3,4,5$. Later on Hardy and Rogers showed in [14] by using the symmetry of the metric function that (1) is equivalent to

$$\rho(Tx),Ty)\le \alpha_1\rho(x,y)+\beta(\rho(Tx,x)+\rho(Ty,x))+\gamma(\rho(Tx,y)+\rho(Ty,x)), \tag{2}$$

where $\alpha=k_1$, $\beta=\frac{k_2+k_3}{2}$ and $\gamma=\frac{k_4+k_5}{2}$.

The Ekeland's variation principle is well known [9, 10, 11, 12]. It has many generalizations and applications in different fields of Mathematics [4, 6, 19].

Let us mention that Ekeland's variational principle holds for any l.s.c maps $T:X\times X\to\mathbb{R}$, provided that X is a partially ordered complete metric space. Unfortunately, when investigating contraction type of maps $F:X\times X\to X$, satisfying the mixed monotone property in a partially ordered complete metric space $X\times X$, the contraction conditions holds only for part of the points $(x,y),(u,v)\in X\times X$. Thus we can not apply Ekeland's variational principle, as it is done in [12]. In [25] Ekeland's variational principle is generalized on classes of subsets of partially ordered complete metric spaces $(X,\preceq)$ with a function $F : X\times X \to X$, which have the mixed monotone property. Later one the idea from [25] is used in [16] to obtain an existence and uniqueness of coupled fixed points for maps with the mixed monotone property in partially ordered metric spaces that satisfy a kind of Chaterjea contraction.

We apply the main result and the proposed technique from [25] to investigate an existence and uniqueness of coupled fixed points for mixed monotone maps of Hardy–Rogers type in partially ordered metric spaces. If in addition every ordered pair of elements from the underlying space, considered as an element of the produced space endowed with the classical sum metric, has an upper or lower bound then the coupled fixed point is unique one.

Thus we generalize the results from [16, 25]

2 Preliminaries

Definition 1. *([3, 13]) Let $(X,\preceq)$ be a partially ordered set and let $F : X\times X \to X$. The function F is said to have the mixed monotone property if*

$$\textit{for any } x_1,x_2,y\in X \textit{ such that } x_1\preceq x_2 \textit{ there holds } F(x_1,y)\preceq F(x_2,y)$$

and

$$\textit{for any } y_1,y_2,x\in X \textit{ such that } y_1\preceq y_2 \textit{ there holds } F(x,y_1)\succeq F(x,y_2).$$

the mixed monotone property of a function $F(\cdot,\cdot)$ means that it is a non-decreasing function of its first variable and it is a non-increasing function of its second variable.

Definition 2. *([3, 13]) Let $F : X\times X \to X$. An ordered pair $(x,y)\in X\times X$ is called coupled fixed point of F if $x = F(x,y)$ and $y = F(y,x)$.*

Let $(X,\rho,\preceq)$ be a partially ordered complete metric space. We endow the product space $X\times X$ with the following partial order $(u,v)\preceq(x,y)$, provided that $x\succeq u$ and $y\preceq v$ holds simultaneously and with the following metric

$$d((x,y),(u,v)) = \rho(x,u)+\rho(y,v)$$

for $(x,y),(u,v)\in X\times X$.

Every where for a partially ordered metric space $(X,\rho,\preceq)$ we will consider the product space $(X\times X,d,\preceq)$ endowed with the mentioned above partial order and metric.

Following [4] an extended real valued function $T : X \to (-\infty,+\infty]$ is called lower semi-continuous (for short l.s.c) if $\{x\in X : f(x) > a\}$ is an open set for each $a\in(-\infty,+\infty]$. Equivalently T is l.s.c if and only if at any point $x_0\in X$ there holds $\liminf\limits_{x\to x_0} f(x)\ge f(x_0)$. A function T is called to be proper function, provided that $T\not\equiv+\infty$.

Just to fit some of the formulas in the text field we will use the notation $u = \left(u^{(1)},u^{(2)}\right)\in X\times X$ and for any $u\in X\times X$ let us denote $\overline{u} = \left(u^{(2)},u^{(1)}\right)$.

Theorem 3. *([25]) Let $(X,\rho,\preceq)$ be a partially ordered complete metric space, $(X\times X,d,\preceq)$ and $F : X\times X\to X$ be a continuous map with the mixed monotone property. Let*

$$V\times V = \{x = (x^{(1)},x^{(2)})\in X\times X : x^{(1)}\preceq F(x) \textit{ and } x^{(2)}\succeq F(\overline{x})\}\neq\emptyset.$$

Let $T : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a proper, l.s.c, bounded from below function. Let $\varepsilon > 0$ be arbitrary chosen and fixed and let $u_0 \in V \times V$ be an ordered pair such that the inequality

$$T(u_0) \leq \inf_{V \times V} T(v) + \varepsilon \tag{3}$$

holds. Then there exists an ordered pair $x \in V \times V$, such that

(i) $T(x) \leq T(u_0) + \varepsilon$

(ii) $d(x, u_0) \leq 1$

(iii) For every $w \in V \times V$ different from $x \in V \times V$ holds the inequality

$$T(w) > T(x) - \varepsilon d(w, x).$$

Proposition 4. *([25]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a map with the mixed monotone property. Let $(x,y) \in X \times X$ satisfies the inequalities $x \preceq F(x,y)$, $y \succeq F(y,x)$ and let us put $u = F(x,y)$ and $v = F(y,x)$. Then there hold $u \preceq F(u,v)$, $v \succeq F(v,u)$, $u \succeq x$ and $v \preceq y$.*

Proposition 5. *([25]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a map with the mixed monotone property. Let $z = (x,y) \in X \times X$ be a coupled fixed point, i.e. $x = F(x,y)$, $y = F(y,x)$ and let (ξ_0, η_0) be comparable with (x,y). Then (ξ_n, η_n) is comparable with $(x,y) = (F(x,y), F(y,x))$ and (η_n, ξ_n) is comparable with $(y,x) = (F(\bar{z}), F(z))$, where $\bar{z} = (y,x)$.*

3 Coupled fixed points for Hardy–Rogers maps with the mixed monotone property in partially ordered metric spaces

Theorem 6. *Let $(X, \rho, \preceq)$ be a partially ordered complete metric space, $(X \times X, d, \preceq)$ and $F : X \times X \to X$ be a continuous map with the mixed monotone property. Let there exists $\alpha + \beta + \gamma \in [0, 1/2)$, so that the inequality*

$$\begin{aligned} \rho(F(x,y), F(u,v)) \quad \leq \quad & \alpha(\rho(x,u) + \rho(y,v)) + \beta(\rho(x, F(x,y) + \rho(u, F(u,v)) \\ & + \gamma(\rho(x, F(u,v)) + \rho(u, F(x,y)). \end{aligned} \tag{4}$$

holds for all $x \succeq u$ and $y \preceq v$. If there exists at least one ordered pair (x,y), such that $x \preceq F(x,y)$ and $y \succeq F(y,x)$, then there exists a coupled fixed point (x,y) of F.

If in addition every pair of elements in $X \times X$ has an lower or an upper bound, then the coupled fixed point is unique.

Proof. It is well known that $a = \frac{2\alpha+\beta+\gamma}{1-\beta-\gamma} \in [0,1)$, provided that $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma \in [0, 1/2)$. Let us consider the function $T : X \times X \to \mathbb{R}$, defined by

$$T(z) = d(z, (F(z), F(\bar{z})) = \rho(x, F(x,y)) + \rho(y, F(y,x)),$$

where $z = (x,y) \in X \times X$. The map T satisfies the conditions of Theorem 3, as far as T is continuous, proper function, bounded from below and the set of all $z \in X \times X$, such that $x \preceq F(z)$ and $y \succeq F(\bar{z})$ is not empty. Let us choose $\varepsilon \in (0, 1-a)$. By Theorem 3 there exists (x,y), satisfying $x \preceq F(x,y)$ and $y \succeq F(y,x)$, such that there holds the inequality

$$T(x,y) \leq T(u,v) + \varepsilon d((x,y),(u,v)) \tag{5}$$

for every $u \preceq F(u,v)$ and $v \succeq F(v,u)$.

Let us put $u = F(x,y)$, $v = F(y,x)$ and $w = (u,v)$. By Proposition 4 it follows that $u \preceq F(u,v)$, $v \succeq F(v,u)$, $u \succeq x$ and $v \preceq y$. From (4) using the symmetry of $\rho(\cdot,\cdot)$ we obtain

$$\begin{array}{rcl} S_{2.1} & = & \rho(F(z),F(F(z),F(\bar{z}))) + \rho(F(\bar{z}),F(F(\bar{z}),F(z))) \\ & = & \rho(F(F(z),F(\bar{z})),F(z)) + \rho(F(\bar{z}),F(F(\bar{z}),F(z))) \\ & = & \rho(F(w),F(z)) + \rho(F(\bar{z}),F(\bar{w})) \\ & \leq & 2\alpha(\rho(u,x)+\rho(v,y)) + \beta(\rho(u,F(w)) + \rho(x,F(z)) + \beta(\rho(v,F(\bar{w})) + \rho(y,F(\bar{z}))) \\ & & +\gamma(\rho(u,u)+\rho(F(w),x)) + \gamma(\rho(v,v)+\rho(F(\bar{w}),y)) \\ & = & 2\alpha(\rho(F(z),x)+\rho(F(\bar{z}),y)) \\ & & +\beta(\rho(F(z),F(F(z),F(\bar{z})) + \rho(x,F(z)) + \beta(\rho(F(\bar{z}),F(F(\bar{z}),F(z)) + \rho(y,F(\bar{z}))) \\ & & +\gamma(\rho(F(F(z),F(\bar{z})),x)) + \rho(F(F(\bar{z}),F(z)),y)) \\ & \leq & 2\alpha(\rho(F(z),x)+\rho(F(\bar{z}),y)) \\ & & +\beta(\rho(F(z),F(F(z),F(\bar{z})) + \rho(x,F(z)) + \beta(\rho(F(\bar{z}),F(F(\bar{z}),F(z)) + \rho(y,F(\bar{z}))) \\ & & +\gamma(\rho(F(F(z),F(\bar{z})),F(z))) + \rho(F(F(\bar{z}),F(z)),F(\bar{z}))) + \gamma(\rho(F(z),x)+\rho(F(\bar{z}),y)) \end{array}$$

and consequently

$$S_{2.1} \leq \frac{2\alpha+\beta+\gamma}{1-\beta-\gamma}(\rho(F(x,y),x)+\rho(F(y,x),y)). \tag{6}$$

From (5) and (6) we obtain

$$\begin{array}{rcl} T(w) & = & d(w,(F(w),F(\overline{w})) \leq \frac{2\alpha+\beta+\gamma}{1-\beta-\gamma}(\rho(x,F(x,y))+\rho(y,F(y,x))) \\ & = & \frac{2\alpha+\beta+\gamma}{1-\beta-\gamma}T(x,y) = aT(x,y). \end{array} \tag{7}$$

Consequently from (5) using (7) we get

$$T(x,y) \leq T(u,v) + \varepsilon d((x,y),(u,v)) \leq aT(x,y) + \varepsilon T(x,y) = (a+\varepsilon)T(x,y).$$

From the choice of $\varepsilon \in (0, 1-a)$ we obtain $T(x,y) < T(x,y)$. From the last inequality it follows that $T(x,y) = d((x,y),(F(x,y),F(y,x))) = 0$, i.e.

$$\rho(x,F(x,y)) + \rho(y,F(y,x)) = 0.$$

Therefore (x,y) is a coupled fixed points of F.

It remains to prove the uniqueness if the additional condition holds. Let there are two coupled fixed points $(x,y),(u,v) \in X \times X$, then $x = F(x,y)$, $y = F(y,x)$, $u = F(u,v)$ and $v = F(v,u)$. By the assumption that any element has an lower or an upper bound it follows from Proposition 4 [13, 20] that there exists (ξ_0,η_0) comparable with (x,y) and (u,v). From Proposition 5 it follows that (ξ_n,η_n) is comparable with both $(x,y) = (F(x,y),F(y,x))$ and $(u,v) = (F(u,v),F(v,u))$ and (η_n,ξ_n) is comparable with both (y,x) and (v,u).

We will apply inequality (4). If $(\xi_n,\eta_n) \succeq (x,y)$, then it satisfies the assumptions of the theorem.

If $(\xi_n,\eta_n) \preceq (x,y)$, using the symmetry of the metrics ρ we get

$$\rho(F(\xi_n,\eta_n),F(x,y)) = \rho(F(x,y),F(\xi_n,\eta_n)) \leq \alpha\rho(x,F(\xi_n,\eta_n)) + \alpha\rho(\xi_n,F(x,y)).$$

Thus we can apply (4) when (ξ_n,η_n) is comparable with $(F(x,y),F(y,x))$.

There exists $n_0 \in \mathbb{N}$, such that the inequality

$$a^{n_0} < \frac{\rho(\xi_0,x)+\rho(\eta_0,y)+\rho(\xi_0,u)+\rho(\eta_0,v)}{\rho(x,u)+\rho(y,v)}$$

holds true.

Use inequality (4) we get that

$$\begin{array}{rcl} S_{2.2} & = & \rho(\xi_n,x)+\rho(\eta_n,y) \\ & = & \rho(F(\xi_{n-1},\eta_{n-1}),F(x,y))+\rho(F(\eta_{n-1},\xi_{n-1}),F(y,x)) \\ & \leq & 2\alpha(\rho(\xi_{n-1},x)+\rho(\eta_{n-1}))+\beta(\rho(\xi_{n-1},F(\xi_{n-1},\eta_{n-1}))+\rho(x,F(x,y))) \\ & & +\beta(\rho(\eta_{n-1},F(\eta_{n-1},\xi_{n-1}))+\rho(y,F(y,x))) \\ & & +\gamma\rho(\xi_{n-1},F(x,y))+\rho(x,F(\xi_{n-1},\eta_{n-1}))+\gamma\rho(\eta_{n-1},F(y,x))+\rho(y,F(\eta_{n-1},\xi_{n-1})) \\ & \leq & 2\alpha(\rho(\xi_{n-1},x))+\rho(\eta_{n-1}))+\beta(\rho(\xi_{n-1},\xi_n)+\rho(\eta_n,\xi_{n-1})) \\ & & +\gamma(\rho(\xi_{n-1},x)+\rho(x,\xi_n)+\rho(\eta_{n-1},y)+\rho(y,\eta_n)) \\ & \leq & 2\alpha(\rho(\xi_{n-1},x))+\rho(\eta_{n-1}))+\beta(\rho(\xi_{n-1},x)+\rho(x,\xi_n)+\rho(\eta_n,y)+\rho(y,\xi_{n-1})) \\ & & +\gamma(\rho(\xi_{n-1},x)+\rho(x,\xi_n)+\rho(\eta_{n-1},y)+\rho(y,\eta_n)) \end{array}$$

and

$$\begin{array}{rcl} \rho(\xi_n,x)+\rho(\eta_n,y) & \leq & \frac{2\alpha+\beta+\gamma}{1-\beta-\gamma}(\rho(\eta_{n-1},y)+\rho(y,\eta_{n-1}) \\ & = & a(\rho(\eta_{n-1},y)+\rho(y,\eta_{n-1}))\leq a^n(\rho(\xi_0,x)+\rho(\eta_0,y)). \end{array}$$

Then we obtain

$$\begin{array}{rcl} \rho(x,u)+\rho(y,v) & \leq & \rho(x,\xi_{n_0})+\rho(\xi_{n_0},u)+\rho(y,\eta_{n_0})+\rho(\eta_{n_0},v) \\ & \leq & a^{n_0}(\rho(\xi_0,x)+\rho(\eta_0,y)+\rho(\xi_0,u)+\rho(\eta_0,v)) \\ & < & \rho(x,u)+\rho(y,v), \end{array}$$

which is a contradiction and that $(x,y)=(u,v)$. □

4 Applications and consequences of Theorem 6

If we take $\beta=\gamma=0$ in Theorem 3 we get the result from [25]. If we take $\alpha=\gamma=0$ in Theorem 3 we get the result from [25]. If we take $\alpha=\beta=0$ in Theorem 3 we get a result from [16].

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