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COMMUTING NONSELFADJOINT OPERATORS AND WAVE EQUATIONS*

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ABSTRACT: *In this paper we present results concerning commuting nonselfadjoint operators and corresponding generalized open systems and matrix wave equations, using Livšic nonselfadjoint operator theory. These results describe the existence and uniqueness of the solutions of the boundary value problem for solutions of the matrix wave equations for output of the generalized open system in the case of n-tuples of commuting nonselfadjoint bounded operators, when one of them belongs to a larger class of nonselfadjoint nondissipative operators.*

KEYWORDS: *Dissipative operator, Operator colligation, Triangular model, Coupling, Open system, Matrix wave equation.*

2010 Math. Subject Classification: *47A48, 35M12*

1 Preliminary results

In this section we will present some preliminary results concerning commuting nonselfadjoint operators generating generalized open systems, corresponding collective motions, and matrix wave equations (considered by G.S. Borisova and K.P. Kirchev in [4]). These essential results are obtained in the case of n-tuples of commuting nonselfadjoint bounded operators when one of them belongs to a larger class of nondissipative operators, presented as couplings of dissipative and antidissipative operators and they are obtained in [3].

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Let us consider commuting nonselfadjoint bounded linear operators $A_1, \dots, A_n$ ($A_k A_s = A_s A_k$, $k = 1, \dots, n$) in a Hilbert space H. Let these operators be embedded in a *commutative regular colligation*

$$X = (A_1, \dots, A_n; H, \Phi, E; \sigma_1, \dots, \sigma_n, \{\gamma_{ks}\}, \{\widetilde{\gamma}_{ks}\}, \\ k, s = 1, \dots, n) \tag{1}$$

where E is another Hilbert space, Φ is a bounded linear mapping of H into E, $\sigma_1, \sigma_2, \dots, \sigma_n$, $\{\gamma_{ks}\}$, $\{\widetilde{\gamma}_{ks}\}$, $(k, s = 1, 2, \dots, n)$ are bounded linear selfadjoint operators in E (where $\gamma_{ks} = \gamma_{ks}^* = -\gamma_{sk}$) and they satisfy the next conditions:

$$(A_k - A_k^*)/i = \Phi^* \sigma_k \Phi, \tag{2}$$

$$\sigma_s \Phi A_k^* - \sigma_k \Phi A_s^* = \gamma_{ks} \Phi, \tag{3}$$

$$\widetilde{\gamma}_{ks} = \gamma_{ks} + i(\sigma_k \Phi \Phi^* \sigma_s - \sigma_s \Phi \Phi^* \sigma_k) \tag{4}$$

for $k, s = 1, 2, \dots, n$. Instead of the term "regular colligation" one can use the term "vessel", that has been coined in [11].

In what follows, we assume that $\dim E < \infty$ (which includes the most of important cases) and $\bigcap\limits_{k=1}^{n} \ker \sigma_k = \{0\}$. If $range\ \Phi = E$, the colligation X is called a *strict colligation*.

The system-theoretic interpretation of n-operator colligation leads to an open n-dimensional system. We consider the generalized open system (introduced by G.S. Borisova and K.P. Kirchev in [4]) from the form

$$\begin{cases} i \frac{1}{\varepsilon_k} \frac{\partial}{\partial x_k} f(x) + A_k f(x) = \Phi^* \sigma_k u(x), \quad k = 1, 2, \dots, n, \\ v(x) = u(x) - i \Phi f(x), \end{cases} \tag{5}$$

where $x = (x_1, \dots, x_n)$, $f(x)|_{\Gamma_+} = f_0(x)$ ($\Gamma_+ = \partial\mathbb{R}^n_+$), $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{C}$ are constants and the vector functions

$$u(x) = u(x_1, x_2, \dots, x_n), \;\; v(x) = v(x_1, x_2, \dots, x_n),$$
$$f(x) = f(x_1, x_2, \dots, x_n)$$

are the input, the output, and the internal state of the open system (5). (In the cases, considered by M.S. Livšic, the open systems are when $\varepsilon_1 = \dots = \varepsilon_n = 1$.)

Direct calculations show ([4], Theorem 6.1) that the system (5) has a solution if the function $f_0(x)$ on Γ_+ satisfies the equations from (5) and the vector function $u(x)$ is a solution of the system

$$\sigma_k \left(-i\frac{1}{\varepsilon_s}\frac{\partial u}{\partial x_s} \right) - \sigma_s \left(-i\frac{1}{\varepsilon_k}\frac{\partial u}{\partial x_k} \right) + \gamma_{sk} u = 0, \tag{6}$$

$(k, s = 1, 2, \dots, n)$, i.e. following M.S. Livšic and Y. Avishai [12] $u(x)$ satisfies the matrix wave equations (6).

The system (5) is over determined in the case when $n \geq 3$. To avoid this one has to consider the additional conditions for the operators $\{\sigma_k, \gamma_{ks}\}$, $k, s = 1, 2, \dots, n$, when $\det \sigma_n \neq 0$. These conditions have been introduced by V. Zolotarev in the paper [14] and have the form:

$$\begin{gathered} \sigma_n^{-1}\sigma_k\sigma_n^{-1}\sigma_s = \sigma_n^{-1}\sigma_s\sigma_n^{-1}\sigma_k, \\ \sigma_n^{-1}\gamma_{kn}\sigma_n^{-1}\gamma_{sn} = \sigma_n^{-1}\gamma_{sn}\sigma_n^{-1}\gamma_{kn}, \\ \sigma_n^{-1}\sigma_k\sigma_n^{-1}\gamma_{sn} + \sigma_n^{-1}\gamma_{kn}\sigma_n^{-1}\sigma_s = \\ = \sigma_n^{-1}\gamma_{kn}\sigma_n^{-1}\gamma_{sn} + \sigma_n^{-1}\gamma_{sn}\sigma_n^{-1}\gamma_{kn} \end{gathered} \tag{7}$$

$k, s = 1, 2, \dots, n-1$. The conditions (7) follow from the equalities of the mixed partial derivatives $\frac{\partial^2 u}{\partial x_k \partial x_s} = \frac{\partial^2 u}{\partial x_s \partial x_k}$, $k, s = 1, 2, \dots, n$. Then from (6) it follows that

$$\gamma_{ks} = \sigma_s\sigma_n^{-1}\gamma_{kn} - \sigma_k\sigma_n^{-1}\gamma_{sn}, \quad k, s = 1, 2, \dots, n. \tag{8}$$

Consequently, when σ_n is invertible matrix, the commutative regular colligation is determined by the matrices $\{\sigma_k\}_1^n$, $\{\gamma_{kn}\}_{k=1}^{n-1}$, satisfying the conditions (7) and other operators γ_{ks}, $k, s = 1, 2, \dots, n$ are defined by the equalities (8) (see [4]).

The selfadjoint operators $\widetilde{\gamma}_{ks}$, defined by (4), satisfy analogous relations as γ_{ks} ($k, s = 1, 2, \dots, n$), i.e.

$$\widetilde{\gamma}_{ks} = \widetilde{\gamma}_{ks}^* = -\widetilde{\gamma}_{sk}, \qquad \sigma_k \Phi A_s - \sigma_s \Phi A_k = \widetilde{\gamma}_{ks} \Phi,$$

$$\sigma_n^{-1} \sigma_k \sigma_n^{-1} \widetilde{\gamma}_{sn} + \sigma_n^{-1} \widetilde{\gamma}_{kn} \sigma_n^{-1} \sigma_s = \sigma_n^{-1} \sigma_s \sigma_n^{-1} \widetilde{\gamma}_{kn} + \sigma_n^{-1} \widetilde{\gamma}_{sn} \sigma_n^{-1} \sigma_k,$$

$$\sigma_n^{-1} \widetilde{\gamma}_{kn} \sigma_n^{-1} \widetilde{\gamma}_{sn} = \sigma_n^{-1} \widetilde{\gamma}_{sn} \sigma_n^{-1} \widetilde{\gamma}_{kn}, \qquad \widetilde{\gamma}_{ks} = \sigma_s \sigma_n^{-1} \widetilde{\gamma}_{kn} - \sigma_k \sigma_n^{-1} \widetilde{\gamma}_{sn}$$

Now in the case when $n \geq 3$ we consider the colligation from the form

$$X = (A_1, \dots, A_n; H, \Phi, E; \sigma_1, \dots, \sigma_n, \{\gamma_{kn}\}, \{\widetilde{\gamma}_{kn}\}, \\ k = 1, 2, \dots, n-1) \tag{9}$$

instead of the commuting regular colligation (1).

Next, if the input $u(x)$ of the generalized open system (5), corresponding to the commutative regular colligation (1), satisfies the matrix wave equations (6), then the output $v(x)$ from (5) satisfies the system (or matrix wave equations)

$$\sigma_k \left(-i \frac{1}{\varepsilon_s} \frac{\partial v}{\partial x_s} \right) - \sigma_s \left(-i \frac{1}{\varepsilon_k} \frac{\partial v}{\partial x_k} \right) + \widetilde{\gamma}_{ks} v = 0, \tag{10}$$

$k, s = 1, 2, \dots, n$ (see [4], Theorem 6.2).

Let us consider the case when one of the operators $A_1, A_2, \dots, A_n$ is a coupling of dissipative and antidissipative operators with real absolutely continuous spectra (for example, A_1) and $\varepsilon_1 = 1$. (The operator B is said to be a couling in a Hilbert space H, if B has the representation

$$B = P_1 B P_1 + P_1 B P_2 + P_2 B P_2, \tag{11}$$

where P_1, P_2 are orthogonal projectors in H) Without loss of generality we can suppose that $A_1 = B$, where B is the triangular model of this coupling (introduced by G.S. Borisova in [1] and investigated by G.S. Borisova and K.P. Kirchev in [6], [7]):

$$
\begin{aligned}
(12)\qquad & Bf(w) = \alpha(w)f(w) - i\int_{a'}^{w} f(\xi)\Pi(\xi)S^*\Pi^*(w)d\xi + \\
& + i\int_{w}^{b'} f(\xi)\Pi(\xi)S\Pi^*(w)d\xi + i\int_{a'}^{w} f(\xi)\Pi(\xi)L\Pi^*(w)d\xi,
\end{aligned}
$$

where $f = (f_1, f_2, \ldots, f_p) \in H = \mathbf{L}^2(\Delta;\mathbb{C}^p)$, $\Delta = [a', b']$, $L : \mathbb{C}^m \longrightarrow \mathbb{C}^m$, $\det L \neq 0$, $L^* = L$, $L = J_1 - J_2 + S + S^*$,

$$
(13)\qquad J_1 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},\ J_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix},\ S = \begin{pmatrix} 0 & 0 \\ \widehat{S} & 0 \end{pmatrix},
$$

r is the number of positive eigenvalues and $m - r$ is the number of negative eigenvalues of the matrix L, $\Pi(w)$ is a measurable $p \times m$ $(1 \leq p \leq m)$ matrix function on Δ, whose rows are linearly independent at each point of a set of positive measure, the matrix function $\widetilde{\Pi}(w) = \Pi^*(w)\Pi(w)$ satisfies the conditions $tr\ \widetilde{\Pi}(w) = 1$, $\widetilde{\Pi}(w)J_1 = J_1\widetilde{\Pi}(w)$, $||\widetilde{\Pi}(w_1) - \widetilde{\Pi}(w_2)|| \leq C|w_1 - w_2|^{\alpha_1}$ for all $w_1, w_2 \in \Delta$ for some constant $C > 0$, α_1 is an appropriate constant with $0 < \alpha_1 \leq 1$ (see [6]), (where $||\ \ ||$ is the norm in $\mathbb{C}^m$) and the function $\alpha : \Delta \longrightarrow \mathbb{R}$ satisfies the conditions:

(i) the function $\alpha(w)$ is continuous strictly increasing on Δ;

(ii) the inverse function $\sigma(u)$ of $\alpha(w)$ is absolutely continuous on $[a, b]$ $(a = \alpha(a'), b = \alpha(b'))$;

(iii) the function $\sigma'(u)$ is continuous and satisfies the relation $|\sigma'(u_1) - \sigma'(u_2)| \leq C|u_1 - u_2|^{\alpha_2}$, $(0 < \alpha_2 \leq 1)$ for all $u_1, u_2 \in [a, b]$ and for some constant $C > 0$.

The operator B from (12) satisfies the condition $(B - B^*)/i = \Phi^* L\Phi$, where the operator $\Phi : H \longrightarrow H$ is defined by the equality

$$
\Phi f(w) = \int_{a'}^{b'} f(w)\Pi(w)dw
$$

and B has the representation (11), $P_1 f(w) = f(w)\Pi(w)J_1 Q(w)$ and $P_2 f(w) = f(w)\Pi(w)J_2 Q(w)$, where $Q(w)$ is $m \times p$ smooth matrix function on Δ satisfying the condition $\Pi(w)Q(w) = I$.

The existence of the wave operators

$$W_\pm(B^*, B) = s - \lim_{x \to \pm\infty} e^{ixB^*} e^{-ixB}$$

of the couple of operators (B, B^*) as strong limits has been established and their explicit form has been obtained in [6] and [2], i.e.

$$W_\pm(B^*, B) = s - \lim_{x \to \pm\infty} e^{ixB^*} e^{-ixB} = \widetilde{S}^*_\mp \widetilde{S}_\mp. \tag{14}$$

The explicit form of the operators $\widetilde{S}_\mp$ on the right hand side of the relation (14) for the operator B with triangular model (12) has been obtained in [6] in the terms of the multiplicative integrals and the finite dimensional analogue of the classical gamma function (introduced in [6]) and presented by (16).

To avoid complications of writing we consider the case when $\alpha(w) = w$, i.e. the operator B has the form

$$\begin{gathered} Bf(w) = wf(w) - i\int_{a'}^{w} f(\xi)\Pi(\xi)S^*\Pi^*(w)d\xi + \\ +i\int_{w}^{b'} f(\xi)\Pi(\xi)S\Pi^*(w)d\xi + i\int_{a'}^{w} f(\xi)\Pi(\xi)L\Pi^*(w)d\xi, \end{gathered} \tag{15}$$

Then the operators $\widetilde{S}_\mp$ take the form (see, for example, [6], [4]).

$$\widetilde{S}_\pm f(w) = (\widehat{S}_\pm f(w))T_\pm, \quad \widehat{S}_\pm f = \widetilde{S}_{11} f + \widetilde{S}_{22} f + \widetilde{S}^\pm_{12} f, \tag{16}$$

$$S_\pm f(w) = (\widehat{S}_\pm f(w))T_\pm \Pi(w)(J_1 |t|^{i\widetilde{\Pi}_1(w)} J_1 + J_2 |t|^{-i\widetilde{\Pi}_2(w)} J_2)Q(w),$$

$$\begin{gathered} T_\pm h = h(J_1 U_{2a'}(w) w^{i\widetilde{\Pi}_1(w)} e^{\mp\frac{\pi}{2}\widetilde{\Pi}_1(w)} \mathbf{\Gamma}^{-1}(I + i\widetilde{\Pi}_1(w))J_1 + \\ +J_2 \widetilde{U}_{2a'}(w) w^{-i\widetilde{\Pi}_2(w)} e^{\pm\frac{\pi}{2}\widetilde{\Pi}_2(w)} \mathbf{\Gamma}^{-1}(I - i\widetilde{\Pi}_2(w))J_2)\Pi^*(w) \end{gathered}$$

$(\forall h \in \mathbb{C}^m)$,

$$\widetilde{S}_{kk} f(w) = \int\limits_{a'}^{x} \widetilde{f}'(\xi) \int\limits_{a'}^{\overleftarrow{w}} e^{\frac{(-1)^{k+1} i\widetilde{\Pi}_1(v)}{v-\xi} dv} d\xi J_k,$$
$$\widetilde{S}_{12}^{\pm} f(w) = -\int\limits_{a'}^{b'} \widetilde{f}'(\xi) \widetilde{F}_{\xi}^{\mp}(w, b') d\xi S,$$
$$U_{2\xi}(w) = \lim_{\delta \to 0} \int\limits_{\xi}^{\overrightarrow{w-\delta}} e^{\frac{-i\widetilde{\Pi}_1(v)}{v-w} dv} e^{i \int\limits_{\xi}^{w-\delta} \frac{\widetilde{\Pi}_1(w)}{v-w} dv},$$
$$\widetilde{U}_{2\xi}(w) = \lim_{\delta \to 0} \int\limits_{\xi}^{\overrightarrow{w-\delta}} e^{\frac{i\widetilde{\Pi}_2(v)}{v-w} dv} e^{-i \int\limits_{\xi}^{w-\delta} \frac{\widetilde{\Pi}_2(w)}{v-w} dv},$$
$$\mathbf{\Gamma}(\varepsilon I - iT(u)) = \int\limits_{0}^{\infty} e^{-x} e^{((\varepsilon-1)I - iT(u)) \ln x} dx \quad (\varepsilon > 0).$$
$$\widetilde{\Pi}_k(w) = J_k \widetilde{\Pi}(w) J_k = J_k \Pi^*(w) \Pi(w) J_k,$$
$$\widetilde{f}(w) = f(w) Q^*(w),$$

$k = 1, 2$. In the last equality, $m \times p$ matrix function $Q(w)$ is smooth on Δ and satisfies the condition $\Pi(w)Q(w) = I$.

2 The main results

Let us consider commuting nonselfadjoint bounded linear operators $A_1, \ldots, A_n$ with $A_1 = B$, B from the form (15) and let they be embedded in the regular colligation

$$X = (A_1 = B, \ldots, A_n; H = \mathbf{L}^2(\Delta, \mathbb{C}^p), \Phi, E = \mathbb{C}^m; \{\sigma_k\}, \{\widetilde{\gamma}_{kn}\}, \\ k = 1, \ldots, n-1).$$

(Here the operators Φ, $\{\sigma_k\}$, $\{\widetilde{\gamma}_{kn}\}$, $k = 1, \ldots, n-1$, are stated as in Section 1.)

Let $\widehat{H}$ be the principal subspace of the colligation X from the form (1) and $A_1 = B$, where B is the triangular model (12), i.e.

$$\widehat{H} = \overline{\mathit{span}}\, \{A_1^{m_1} \ldots A_n^{m_n} \Phi^* E, m_1, \ldots, m_n \in \mathbb{N} \cup \{0\}\}. \tag{17}$$

Let $\widetilde{H}$ be the set of solutions of the system (10)

$$v_h(x_1,\ldots,x_n) = \Phi e^{i(\varepsilon_1 x_1 A_1+\cdots+\varepsilon_n x_n A_n)}h, \quad h\in\widehat{H}. \tag{18}$$

Let the operator $U:\widehat{H}\longrightarrow\widetilde{H}$ be defined by the equality

$$Uh = \Phi e^{i(\varepsilon_1 x_1 A_1+\cdots+\varepsilon_n x_n A_n)}h = v_h(x_1,x_2,\ldots,x_n), \quad h\in\widehat{H}. \tag{19}$$

From Theorem 6 [3] it follows that the equality

$$\begin{gathered}\langle v_{h_1}(x_1,\ldots,x_n), v_{h_2}(x_1,\ldots,x_n)\rangle = \\ = \lim_{x_1\to+\infty}(e^{ix_1A_1}h_1, e^{ix_1A_1}h_2)+ \\ + \int_0^\infty (\sigma_n v_{h_1}(x_1,0,\ldots,0), v_{h_2}(x_1,0,\ldots,0))dx_1 = \\ = (\widetilde{S}_+^*\widetilde{S}_+h_1,h_2)+ \\ + \int_0^\infty (\sigma_1 v_{h_1}(x_1,0,\ldots,0), v_{h_2}(x_1,0,\ldots,0))dx_1,\end{gathered} \tag{20}$$

defines a scalar product in the space $\widetilde{H}$ of solutions $v_h(x)$ from (18) ($h\in\widehat{H}$, $\varepsilon_1=1$, $A_1=B$).

The equalities (20) and $\langle v_{h_1}, v_{h_2}\rangle = (h_1,h_2)$ imply also that the operator U, defined by (19), is an isometric one and $\langle v_{h_1}, v_{h_2}\rangle = \langle Uh_1, Uh_2\rangle = (h_1,h_2), h_1,h_2\in\widehat{H}$.

Following the introduced terminology by M.S. Livšic in [8] the functions $v_h(x_1,\ldots,x_n)$ and $v_h(x_1,0,\ldots,0)$ are said to be the output representation and the mode of the element $h\in\widetilde{H}$ correspondingly.

The case of two commuting nonselfadjoint operators (A_1,A_2) where A_1 is a dissipative operator (i.e. $(A_1-A_1^*)/i\geq 0$) with zero limit $\lim\limits_{x_1\to\infty}(e^{ix_1A_1}h, e^{ix_1A_1}h) = 0$ ($h\in H$) (i.e. the characteristic operator function $W(1,0,z) = W_{A_1}(z)$ is an inner function), $\varepsilon_1 = \varepsilon_2 = 1$, considered by M.S. Livšic in [8], and the case of n commuting nonselfadjoint operators $(A_1,\ldots,A_n)$, where A_1 is a dissipative operator with nonzero limit $\lim\limits_{x_1\to\infty}(e^{ix_1A_1}h, e^{ix_1A_1}h)\neq 0$ ($h\in$

H), considered by G.S. Borisova and K.P. Kirchev in [5], show that a given mode $v_0(x_1)$ determines uniquely the corresponding output representation $v(x_1,\dots,x_n)$ by the equations (10) and $v(x_1,0,\dots,0) = v_0(x_1)$ in the region of an existence and uniqueness of the solutions (see [13]). The case when $n = 2$, $\varepsilon_1 = \varepsilon_2 = 1$ for operators A_1, A_2 with $\det\sigma_2 \neq 0$ and with an assumption for an existence of the limit $\lim\limits_{x_1\to\infty}(e^{ix_1A_1}h, e^{ix_1A_1}h)$ has been considered in [11].

The next theorem solves a similar problem for the output and the mode

$$v_h(x_1,\dots,x_n), \quad v_h(0,\dots,0,x_n)$$

correspondingly in the case of n operators ($n \geq 3$) with nonzero constants $\varepsilon_1,\dots,\varepsilon_n$, when $A_1 = B$ is a coupling of dissipative and antidissipative operators with real absolutely continuous spectra, which ensures the existence of the limit $\lim\limits_{x\to+\infty}(e^{ixB}f, e^{ixB}f)$, obtained explicitly in [6]. In this case we essentialy use the conditions of V.A. Zolotarev [14].

We consider now the boundary value problem for solutions of the partial differential equations

$$\text{(21)} \qquad \begin{cases} \sigma_n\left(-i\frac{1}{\varepsilon_k}\frac{\partial v}{\partial x_k}\right) - \sigma_k\left(-i\frac{1}{\varepsilon_n}\frac{\partial v}{\partial x_n}\right) + \widetilde{\gamma}_{kn}v = 0, \\ (k = 1,2,\dots,n-1) \\ v(0,\dots,0,x_n) = v_0(x_n) \end{cases}$$

which are restrictions to $\mathbb{R}^n$ of entire functions on $\mathbb{C}^n$. We will denote by $(z_1,\dots,z_n)$ the coordinates on $\mathbb{C}^n$ and by $(x_1,\dots,x_n)$ the coordinates on $\mathbb{R}^n$.

Theorem 1. *Let* $\sigma_1,\sigma_2,\dots,\sigma_n$, $\{\widetilde{\gamma}_{kn}\}$ *(*$k = 1,2,\dots,n-1$*) be* $m\times m$ *hermitian matrices with* $\det\sigma_n \neq 0$ *and they satisfy the conditions of V.A. Zolotarev (7). Then*

1) if $v(x_1,\dots,x_n)$ *is a solution of*

$$\text{(22)} \quad \sigma_n\frac{1}{\varepsilon_k}\frac{\partial v}{\partial x_k} - \sigma_k\frac{1}{\varepsilon_n}\frac{\partial v}{\partial x_n} + i\widetilde{\gamma}_{kn}v = 0, \quad k = 1,2,\dots,n-1,$$

which is a restriction to $\mathbb{R}^n$ of entire function on $\mathbb{C}^n$, and $v(0,\dots,0,x) = 0$ for all $x \in \mathbb{R}$, then $v(x_1,\dots,x_n) = 0$;

2) if $\{v_l(x_1,\dots,x_n)\}$ is a sequence of solutions of (22) which are restrictions to $\mathbb{R}^n$ of entire function on $\mathbb{C}^n$, satisfying the condition $v_l(x,0,\dots,0) \longrightarrow g(x)$ as $l \to \infty$ ($\forall x \in \mathbb{R}$), where $g(x)$ is a function on $\mathbb{R}$ which is infinitely differentiable in a neighbourhood of 0 and there exists a constant C such that

$$\lim_{l\to\infty}\left(\left(\frac{\partial v_l}{\partial x_n^k}(0,\dots,0) - \frac{d^k g}{dx^k}(0)\right)/C^k\right) = 0 \tag{23}$$

uniformly according to k, then there exists a solution $v(x_1,\dots,x_n)$ of (22) which is a restriction to $\mathbb{R}^n$ of an entire function on $\mathbb{C}^n$, such that $v(0,\dots,0,x) = g(x)$ for all $x \in \mathbb{R}$ and the sequence $v_l(z_1,\dots,z_n) \longrightarrow v(z_1,\dots,z_n)$ as $l \to \infty$ uniformly on compact subset on $\mathbb{C}^n$.

Proof. At first we will prove *1)*. Let $v(x_1,\dots,x_n)$ be a solution of (22) and a restriction to $\mathbb{R}^n$ of an entire function on $\mathbb{C}^n$. Let

$$v(0,\dots,0,x) = 0 \qquad \forall x \in \mathbb{R}. \tag{24}$$

Now we have

$$\frac{\partial v}{\partial x_n}(0,\dots,0,x) = 0. \tag{25}$$

Then from (22) we obtain that

$$\begin{gathered}\tfrac{1}{\varepsilon_k}\sigma_n\tfrac{\partial v}{\partial x_k}(0,\dots,0,x) - \tfrac{1}{\varepsilon_n}\sigma_k\tfrac{\partial v}{\partial x_n}(0,\dots,0,x) + \\ +i\widetilde{\gamma}_{kn}v(0,\dots,0,x) = 0\end{gathered} \tag{26}$$

($\forall k = 1,2,\dots,n-1$). The equalities (24), (25) and the condition $\det\sigma_n \neq 0$ imply that

$$\frac{\partial v}{\partial x_k}(0,\dots,0,x) = 0 \quad \forall k = 1,2,\dots,n-1, \quad \forall x \in \mathbb{R}. \tag{27}$$

Now from (25) and (22) we obtain that

$$\frac{\partial^{r_n} v}{\partial x_n^{r_n}}(0,\dots,0,x)=0 \quad \forall r_n=1,2,\dots, \quad \forall x\in\mathbb{R} \tag{28}$$

and

$$\frac{1}{\varepsilon_k}\sigma_n\frac{\partial^{r_n} v}{\partial x_n^{r_n-1}\partial x_k}-\frac{1}{\varepsilon_n}\sigma_k\frac{\partial^{r_n} v}{\partial x_n^{r_n}}+i\widetilde{\gamma}_{kn}\frac{\partial^{r_n-1} v}{\partial x_n^{r_n-1}}=0. \tag{29}$$

Using (28), (24) and $\det\sigma_n\neq 0$ the equality (29) takes the form

$$\frac{\partial^{r_n} v}{\partial x_n^{r_n-1}\partial x_k}(0,\dots,0,x)=0 \quad \forall r_n=1,2,\dots, \quad \forall x\in\mathbb{R}. \tag{30}$$

From the equality (29) after differentiating with respect to x_k we obtain

$$\sigma_n\frac{1}{\varepsilon_k}\frac{\partial^{r_n+1} v}{\partial x_n^{r_n-1}\partial x_k^2}-\sigma_k\frac{1}{\varepsilon_n}\frac{\partial^{r_n+} v}{\partial x_n^{r_n}\partial x_k}+i\widetilde{\gamma}_{kn}\frac{\partial^{r_n} v}{\partial x_n^{r_n}\partial x_k}=0. \tag{31}$$

Now from (30) and (31) it follows that

$$\frac{\partial^{r_n+1} v}{\partial x_n^{r_n-1}\partial x_k^2}(0,\dots,0,x)=0 \quad \forall r_n=1,2,\dots. \tag{32}$$

Analogously to (32) by induction we obtain that

$$\frac{\partial^{r_n+r_k} v}{\partial x_n^{r_n}\partial x_k^{r_k}}(0,\dots,0,x)=0 \tag{33}$$

$\forall r_n, r_k=1,2,\dots$, $\forall k=1,2,\dots,n-1$.

Now we consider the equality

$$\sigma_n\frac{1}{\varepsilon_1}\frac{\partial v}{\partial x_1}-\sigma_1\frac{1}{\varepsilon_n}\frac{\partial v}{\partial x_n}+i\widetilde{\gamma}_{1n}v=0. \tag{34}$$

Then

$$\sigma_n\frac{1}{\varepsilon_1}\frac{\partial^2 v}{\partial x_2\partial x_1}-\sigma_1\frac{1}{\varepsilon_n}\frac{\partial^2 v}{\partial x_2\partial x_n}+i\widetilde{\gamma}_{1n}\frac{\partial v}{\partial x_2}=0 \tag{35}$$

and hence in the point $(0,\ldots,0,x)$ we obtain that

$$\frac{\partial^2 v}{\partial x_2 \partial x_1}(0,\ldots,0,x)=0 \tag{36}$$

which follows from (33) and (27).

From the equality (26) we have

$$\sigma_n \frac{1}{\varepsilon_k}\frac{\partial^2 v}{\partial x_k^2}-\sigma_k \frac{1}{\varepsilon_n}\frac{\partial^2 v}{\partial x_k \partial x_n}+i\widetilde{\gamma}_{kn}\frac{\partial v}{\partial x_k}=0 \tag{37}$$

and consequently from (30) and (27) it follows that

$$\frac{\partial^2 v}{\partial x_k^2}(0,\ldots,0,x)=0 \quad \forall k=1,2,\ldots,n-1. \tag{38}$$

Now from (37) we obtain that (after differentiating with respect to x_k)

$$\sigma_n \frac{1}{\varepsilon_k}\frac{\partial^3 v}{\partial x_k^3}-\sigma_k \frac{1}{\varepsilon_n}\frac{\partial^3 v}{\partial x_k^2 \partial x_n}+i\widetilde{\gamma}_{kn}\frac{\partial^2 v}{\partial x_k^2}=0. \tag{39}$$

Then (38) and (33) imply that

$$\frac{\partial^3 v}{\partial x_k^3}(0,\ldots,0,x)=0 \quad \forall k=1,2,\ldots,n-1. \tag{40}$$

Analogously by induction we obtain that

$$\frac{\partial^{r_k} v}{\partial x_k^{r_k}}(0,\ldots,0,x)=0 \;\; \forall k=1,2,\ldots,n-1, \;\; \forall r_k=1,2,\ldots. \tag{41}$$

From (35) (after differentiating with respect to x_2) we have

$$\sigma_n \frac{1}{\varepsilon_1}\frac{\partial^3 v}{\partial x_2^2 \partial x_1}-\sigma_1 \frac{1}{\varepsilon_n}\frac{\partial^3 v}{\partial x_2^2 \partial x_n}+i\widetilde{\gamma}_{1n}\frac{\partial^2 v}{\partial x_2^2}=0 \tag{42}$$

and consequently using (38) and (30) it follows that

$$\frac{\partial^3 v}{\partial x_2^2 \partial x_1}(0,\dots,0,x) = 0. \tag{43}$$

Analogously we obtain that

$$\sigma_n \frac{1}{\varepsilon_1} \frac{\partial^{r_2+1} v}{\partial x_2^{r_2} \partial x_1} - \sigma_1 \frac{1}{\varepsilon_n} \frac{\partial^{r_2+1} v}{\partial x_2^{r_2} \partial x_n} + i\widetilde{\gamma}_{1n} \frac{\partial^{r_2} v}{\partial x_2^{r_2}} = 0 \tag{44}$$

and then

$$\frac{\partial^{r_2+1} v}{\partial x_2^{r_2} \partial x_1}(0,\dots,0,x) = 0, \qquad \forall r_2 = 1,2,\dots. \tag{45}$$

Next, after differentiating of (44) with respect to x_n it follows that

$$\sigma_n \frac{1}{\varepsilon_1} \frac{\partial^{r_2+2} v}{\partial x_n \partial x_2^{r_2} \partial x_1} - \sigma_1 \frac{1}{\varepsilon_n} \frac{\partial^{r_2+1} v}{\partial x_2^{r_2} \partial x_n^2} + i\widetilde{\gamma}_{1n} \frac{\partial^{r_2} v}{\partial x_n \partial x_2^{r_2}} = 0 \tag{46}$$

and then

$$\frac{\partial^{r_2+2} v}{\partial x_n \partial x_2^{r_2} \partial x_1}(0,\dots,0,x) = 0, \qquad \forall r_2 = 1,2,\dots. \tag{47}$$

The equality (47) follows from (33).

Analogously from (44) (after differentiating with respect to x_n and induction) we have

$$\sigma_n \frac{1}{\varepsilon_1} \frac{\partial^{r_2+r_n+1} v}{\partial x_n^{r_n} \partial x_2^{r_2} \partial x_1} - \sigma_1 \frac{1}{\varepsilon_n} \frac{\partial^{r_2+r_n+1} v}{\partial x_2^{r_2} \partial x_n^{r_n+1}} + i\widetilde{\gamma}_{1n} \frac{\partial^{r_2+r_n} v}{\partial x_n^{r_n} \partial x_2^{r_2}} = 0 \tag{48}$$

and hence

$$\frac{\partial^{r_2+r_n+1} v}{\partial x_n^{r_n} \partial x_2^{r_2} \partial x_1}(0,\dots,0,x) = 0, \qquad \forall r_2, r_n = 1,2,\dots. \tag{49}$$

After differentiating of (48) with respect to x_1 and using (49) we obtain that

$$\begin{aligned}(50)\qquad &\sigma_n \tfrac{1}{\varepsilon_1} \tfrac{\partial^{r_2+r_n+2}v}{\partial x_n^{r_n}\partial x_2^{r_2}\partial x_1^2} - \sigma_1 \tfrac{1}{\varepsilon_n} \tfrac{\partial^{r_2+r_2+2}v}{\partial x_2^{r_2}\partial x_n^{r_n+1}\partial x_1} + \\ &+ i\widetilde{\gamma}_{1n} \tfrac{\partial^{r_2+r_n+1}v}{\partial x_n^{r_n}\partial x_2^{r_2}\partial x_1} = 0,\end{aligned}$$

and hence

$$(51)\qquad \frac{\partial^{r_2+r_n+2}v}{\partial x_n^{r_n}\partial x_2^{r_2}\partial x_1^2}(0,\dots,0,x) = 0, \qquad \forall r_2, r_n = 1, 2, \dots .$$

Analogously from (50) and (51) we obtain that

$$\begin{aligned}(52)\qquad &\sigma_n \tfrac{1}{\varepsilon_1} \tfrac{\partial^{r_1+r_2+r_n}v}{\partial x_n^{r_n}\partial x_2^{r_2}\partial x_1} - \sigma_1 \tfrac{1}{\varepsilon_n} \tfrac{\partial^{r_n+r_2+r_2+}v}{\partial x_2^{r_2}\partial x_n^{r_n+1}} + \\ &+ i\widetilde{\gamma}_{1n} \tfrac{\partial^{r_2+r_n}v}{\partial x_n^{r_n}\partial x_2^{r_2}\partial x_1^{r_1-1}} = 0\end{aligned}$$

and

$$(53)\qquad \frac{\partial^{r_1+r_2+r_n+}v}{\partial x_1^{r_1}\partial x_2^{r_2}\partial x_n^{r_n}}(0,\dots,0,x) = 0, \qquad \forall r_1, r_2, r_n = 1, 2, \dots .$$

Analogously to the equality (53) it can be obtained that

$$(54)\qquad \frac{\partial^{r_k+r_j+r_n}v}{\partial x_k^{r_k}\partial x_j^{r_j}\partial x_n^{r_n}}(0,\dots,0,x) = 0,$$

$\forall k \neq j, \quad k, j = 1, 2, \dots, n-1, \forall r_k, r_j, r_n = 1, 2, \dots$. Now from the equality (48) (using differentiating with respect to x_3) it follows that

$$\begin{aligned}(55)\qquad &\sigma_n \tfrac{1}{\varepsilon_1} \tfrac{\partial^{r_2+r_n+2}v}{\partial x_3 \partial x_n^{r_n}\partial x_2^{r_2}\partial x_1} - \sigma_1 \tfrac{1}{\varepsilon_n} \tfrac{\partial^{r_2+r_n+2}v}{\partial x_3 \partial x_2^{r_2}\partial x_n^{r_n+1}} + \\ &+ i\widetilde{\gamma}_{1n} \tfrac{\partial^{r_2+r_n}v}{\partial x_3 \partial x_n^{r_n}\partial x_2^{r_2}} = 0\end{aligned}$$

and then using (54) we obtain that

$$(56)\qquad \frac{\partial^{r_2+r_n+2}v}{\partial x_3 \partial x_n^{r_n}\partial x_2^{r_2}\partial x_1}(0,\dots,0,x) = 0.$$

Then analogously to obtaining (55) it follows that

$$\begin{gathered}\sigma_n \tfrac{1}{\varepsilon_1} \tfrac{\partial^{r_3+r_2+r_n+1} v}{\partial x_3^{r_3} \partial x_n^{r_n} \partial x_2^{r_2} \partial x_1} - \sigma_1 \tfrac{1}{\varepsilon_n} \tfrac{\partial^{r_3+r_2+r_n+1} v}{\partial x_3^{r_3} \partial x_2^{r_2} \partial x_n^{r_n+1}} + \\ + i\widetilde{\gamma}_{1n} \tfrac{\partial^{r_3+r_2+r_n} v}{\partial x_3^{r_3} \partial x_n^{r_n} \partial x_2^{r_2}} = 0\end{gathered} \tag{57}$$

and

$$\frac{\partial^{r_2+r_3+r_n+1} v}{\partial x_3^{r_3} \partial x_n^{r_n} \partial x_2^{r_2} \partial x_1}(0, \ldots, 0, x) = 0. \tag{58}$$

Then from (57) we obtain that

$$\frac{\partial^{r_1+r_2+r_3+r_n} v}{\partial x_1^{r_1} \partial x_2^{r_2} \partial x_3^{r_3} \partial x_n^{r_n}}(0, \ldots, 0, x) = 0 \tag{59}$$

$\forall r_1, r_2, r_3, r_n = 1, 2, \ldots$. Analogously and consecutively we obtain that

$$\frac{\partial^{r_1+r_2+\ldots+r_n} v}{\partial x_1^{r_1} \partial x_2^{r_2} \ldots \partial x_n^{r_n}}(0, \ldots, 0, x) = 0 \tag{60}$$

$\forall r_1, r_2, \ldots, r_n = 1, 2, \ldots$.

But $v(x_1, \ldots, x_n)$ is a solution of (22) which is a restriction on $\mathbb{R}^n$ of an entire function on $\mathbb{C}^n$. Then the entire function $v(z_1, \ldots, z_n)$ has the representation

$$\begin{gathered} v(z_1, \ldots, z_n) = \\ = \sum_{r_1 \geq 0, \ldots, r_n \geq 0} \tfrac{1}{r_1! r_2! \ldots r_n!} \tfrac{\partial^{r_1+r_2+\ldots+r_n} v}{\partial z_1^{r_1} \partial z_2^{r_2} \ldots \partial z_n^{r_n}}(0, \ldots, 0, x). \\ .z_1^{k_1} \ldots z_{n-1}^{r_{n-1}} (z_n - x)^{r_n}. \end{gathered} \tag{61}$$

The equality (61) together with (60) implies that

$$v(z_1, \ldots, z_n) = 0, \qquad \forall (z_1, \ldots, z_n) \in \mathbb{C}^n$$

and consequently

$$v(x_1, \ldots, x_n) = 0, \qquad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{62}$$

Now if $v_1(x_1, \ldots, x_n)$ and $v_2(x_1, \ldots, x_n)$ are different solutions of the partial differential equations

$$\begin{aligned} &\sigma_n \tfrac{1}{\varepsilon_k} \tfrac{\partial v}{\partial x_k} - \sigma_k \tfrac{1}{\varepsilon_n} \tfrac{\partial v}{\partial x_n} + i\widetilde{\gamma}_{kn} v = 0, \quad k = 1, 2, \ldots, n-1, \\ &v(0, \ldots, 0, x) = g(x) \end{aligned} \tag{63}$$

i.e. $v_1(x_1, \ldots, x_n)$ and $v_2(x_1, \ldots, x_n)$ satisfy the conditions

$$v_1(0, \ldots, 0, x) = g(x), \qquad v_2(0, \ldots, 0, x) = g(x) \qquad \forall x \in \mathbb{R}$$

then the relation (62) implies that

$$v_1(x_1, \ldots, x_n) = v_2(x_1, \ldots, x_n).$$

Hence if the system (63) has the solution, this solution is unique.

Now we will prove *2)*. Let $\{v_l(x_1, \ldots, x_n)\}$ be a sequence of solutions of (22) satisfying the conditions in *2)*. Using the assumption (23) it follows that for all $\varepsilon > 0$ there exists N such that for all $l, s > N$ the next inequality holds

$$\left\| \frac{\partial^k v_l}{\partial x_n^k}(0, \ldots, 0) - \frac{\partial^k v_s}{\partial x_n^k}(0, \ldots, 0) \right\| < \varepsilon |\varepsilon_n| C^k \tag{64}$$

$\forall k = 1, 2, \ldots$.

After differentiation of the equality (22) r times with respect to x_n we obtain that

$$\sigma_n \frac{1}{\varepsilon_k} \frac{\partial^{r+1} v_l}{\partial x_n^r \partial x_k} - \sigma_k \frac{1}{\varepsilon_n} \frac{\partial^{r+1} v_l}{\partial x_n^{r+1}} + i\widetilde{\gamma}_{kn} \frac{\partial^r v_l}{\partial x_n^r} = 0, \tag{65}$$

($\forall l = 1, 2, \ldots$, $\forall k = 1, 2, \ldots, n-1$). Then from (65) it follows that

$$\begin{aligned} \sigma_n \tfrac{1}{\varepsilon_k} \tfrac{\partial^{rn+1} v_l}{\partial x_n^{rn} \partial x_k} - \sigma_n \tfrac{1}{\varepsilon_k} \tfrac{\partial^{rn+1} v_s}{\partial x_n^{rn} \partial x_k} = \\ = \sigma_k \tfrac{1}{\varepsilon_n} \tfrac{\partial^{rn+1} v_l}{\partial x_n^{rn+1}} - \sigma_k \tfrac{1}{\varepsilon_n} \tfrac{\partial^{rn+1} v_s}{\partial x_n^{rn+1}} - i\widetilde{\gamma}_{kn} \left(\tfrac{\partial^{rn} v_l}{\partial x_n^{rn}} - \tfrac{\partial^{rn} v_s}{\partial x_n^{rn}} \right) = 0 \end{aligned} \tag{66}$$

($\forall k = 1, 2, \ldots, n-1$). But from (64) and (66) we have

$$\begin{aligned}
&\left\| \tfrac{1}{\varepsilon_k} \tfrac{\partial^{r_n+1} v_l}{\partial x_n^{r_n} \partial x_k}(0,\ldots,0) - \tfrac{1}{\varepsilon_k} \tfrac{\partial^{r_n+1} v_s}{\partial x_n^{r_n} \partial x_k}(0,\ldots,0) \right\| = \\
&= \left\| \sigma_n^{-1} \sigma_k \tfrac{1}{\varepsilon_n} \left(\tfrac{\partial^{r_n+1} v_l}{\partial x_n^{r_n+1}}(0,\ldots,0) - \tfrac{\partial^{r_n+1} v_s}{\partial x_n^{r_n+1}}(0,\ldots,0) \right) - \right. \\
&\left. - i\sigma_n^{-1} \widetilde{\gamma}_{kn} \left(\tfrac{\partial^{r_n} v_l}{\partial x_n^{r_n}}(0,\ldots,0) - \tfrac{\partial^{r_n} v_s}{\partial x_n^{r_n}}(0,\ldots,0) \right) \right\| < \\
&< \left\| \sigma_n^{-1} \sigma_k \right\| \tfrac{\varepsilon}{|\varepsilon_n|} C^{r_n+1} + ||\sigma_n^{-1} \widetilde{\gamma}_{kn}|| \varepsilon C^{r_n} = \\
&= \varepsilon \left(\left\| \sigma_n^{-1} \sigma_k \right\| \tfrac{C}{|\varepsilon_n|} + ||\sigma_n^{-1} \widetilde{\gamma}_{kn}|| \right) C^{r_n},
\end{aligned}$$

i.e.

$$\begin{aligned}
&\left\| \tfrac{\partial^{r_n+1} v_l}{\partial x_n^{r_n} \partial x_k}(0,\ldots,0) - \tfrac{\partial^{r_n+1} v_s}{\partial x_n^{r_n} \partial x_k}(0,\ldots,0) \right\| < \\
&< \varepsilon |\varepsilon_k| \left(\left\| \sigma_n^{-1} \sigma_k \right\| \tfrac{C}{|\varepsilon_n|} + ||\sigma_n^{-1} \widetilde{\gamma}_{kn}|| \right) C^{r_n} = \varepsilon C_k C^{r_n},
\end{aligned} \tag{67}$$

($\forall k = 1, 2, \ldots, n-1, \forall r_n = 1, 2, \ldots$), where we have denoted

$$C_k = |\varepsilon_k| \left(\left\| \sigma_n^{-1} \sigma_k \right\| \frac{C}{|\varepsilon_n|} + ||\sigma_n^{-1} \widetilde{\gamma}_{kn}|| \right) \tag{68}$$

We apply consecutively similar considerations and it can be obtain that

$$\begin{aligned}
&\left\| \tfrac{\partial^{r_1+r_2+\ldots+r_n} v_l}{\partial x_1^{r_1} \partial x_2^{r_2} \ldots \partial x_n^{r_n}}(0,\ldots,0) - \tfrac{\partial^{r_1+r_2+\ldots+r_n} v_s}{\partial x_1^{r_1} \partial x_2^{r_2} \ldots \partial x_n^{r_n}}(0,\ldots,0) - \right\| < \\
&< \varepsilon C_1^{r_1} C_2^{r_2} \ldots C_{n-1}^{r_{n-1}} C^{r_n}
\end{aligned} \tag{69}$$

($\forall k = 1, 2, \ldots, n-1$, $\forall r_n = 1, 2, \ldots$), where C_k has the form (68).

But $v_l(z_1,\ldots,z_n)$ is entire function and then for all $l,s>N$ we have (70)

$$\|v_l(z_1,\ldots,z_n)-v_s(z_1,\ldots,z_n)\|=$$
$$=\left\|\sum_{k_1,k_2,\ldots,k_n}\frac{1}{k_1!k_2!\ldots k_n!}\frac{\partial^{k_1+k_2+\ldots+k_n}v_l}{\partial z_1^{k_1}\partial z_2^{k_2}\ldots\partial z_n^{k_n}}(0,\ldots,0,0)z_1^{k_1}\ldots z_{n-1}^{r_{n-1}}z_n^{r_n}-\right.$$
$$\left.-\sum_{k_1,k_2,\ldots,k_n}\frac{1}{k_1!k_2!\ldots k_n!}\frac{\partial^{k_1+k_2+\ldots+k_n}v_s}{\partial z_1^{k_1}\partial z_2^{k_2}\ldots\partial z_n^{k_n}}(0,\ldots,0,0)z_1^{k_1}\ldots z_{n-1}^{r_{n-1}}z_n^{r_n}\right\|\leq$$
$$\leq\sum_{k_1,k_2,\ldots,k_n}\frac{1}{k_1!k_2!\ldots k_n!}\varepsilon C_1^{r_1}C_2^{r_2}\ldots C_{n-1}^{r_{n-1}}C^{r_n}|z_1|^{k_1}|z_2|^{k_2}\ldots|z_n|^{k_n}=$$
$$=\varepsilon e^{C_1|z_1|+C_2|z_2|+\ldots+C_{n-1}|z_{n-1}|+C|z_n|}.$$

Consequently $\{v_l(z_1,\ldots,z_n)\}$ is uniformly convergent sequence on the compact sets in $\mathbb{C}^n$. The theorem is proved. $\square$

If the matrices $\sigma_1,\ldots,\sigma_n$, $\{\gamma_{kn}\}$ are selfadjoint $m\times m$ matrices, satisfying the conditions of V.A. Zolotarev (7), $\{\gamma_{ks}\}$ $(k,s=1,2,\ldots,n-1)$ are defined by the equality (8), and the matrices $\{\widetilde{\gamma}_{ks}\}$ $(k,s=1,2,\ldots,n)$ are defined by (4), the Theorem 1 implies that the solution $v(x_1,\ldots,x_n)$ of the system (22) satisfies the system (10).

In the case when the selfadjoint operators $\{\sigma_k\}_1^n$ and $\{\gamma_{ks}\}$ satisfy the conditions (7) and the operators $\{\widetilde{\gamma}_{ks}\}$ are defined by (4), then the system

$$\begin{cases}\sigma_k\left(-i\frac{1}{\varepsilon_s}\frac{\partial v}{\partial x_s}\right)-\sigma_s\left(-i\frac{1}{\varepsilon_k}\frac{\partial v}{\partial x_k}\right)+\widetilde{\gamma}_{sk}v=0\\ v(0,\ldots,0,x)=g(x),\quad x\in\mathbb{R}\end{cases}$$

$(k,s=1,2,\ldots,n)$ (i.e. (10)) has a unique solution satisfying the condition $v(0,\ldots,0,x)=g(x)$, which is a restriction to $\mathbb{R}^n$ of an entire function on $\mathbb{C}^n$.

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