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ON ISOCLINISM OF CERTAIN NILPOTENCY CLASS 2 p-GROUPS*

IVO M. MICHAILOV, IVAYLO A. DIMITROV, IVAN S. IVANOV

ABSTRACT: *In this report we show that any p-group which is of nipotency class 2 and is an internal product of abelian groups is isoclinic to a semidirect product of abelian groups.*

KEYWORDS: *p-group, isoclinic, semidirect product, nilpotency class 2*

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1 Introduction

In [1] James gives a description in terms of generators and relation of all p-groups of orders $\leq p^6$. The groups in the list are collected together in isoclinism families. Two groups G and H with centers $Z(G)$ and $Z(H)$ and derived (i.e., comutator) subgroups G' and H' are said to be isoclinic (written $G \sim H$) if there exist isomorphisms

$$\begin{aligned} \theta &: G/Z(G) \to H/Z(H), \\ \phi &: G' \to H', \end{aligned}$$

such that $\phi([\alpha,\beta]) = [\alpha',\beta']$ for all $\alpha,\beta \in G$, where
$\alpha' Z(H) = \theta(\alpha Z(G)))$ and $\beta' Z(H) = \theta(\beta Z(G))$. It is easy to show that this relation is well defined and is in fact an equivalence relation. The pair (θ,ϕ) is called and isoclinism.

Next, recall that a group G is said to be of nilpotency class 2 if $G' \leq Z(G)$. Assume that a group G of nilpotency class 2 is an internal product of two abelian subgroups A and B. Based on the definition of isoclinism, we arrive to the hypothesis that G is isoclinic to a semidirect

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product of two abelian groups: $G \sim F = \tilde{A} \rtimes \tilde{B}$. The technical verification of this claim however is rather long and complicated. The main idea is to define the group F such that it removes the obstacle for an internal product to be a semidirect product and then define and isoclinism between G and F. We will illustrate this process in the following section for a group G with 4 generators.

2 On isoclinism of a p-group of nipotency class 2 which is an internal product of two abelian groups

Let G be a p-group of nilpotency class 2, let B be an abelian normal subgroup with two generators, and let the quotient group G/B be an abelian group with two generators. We can write G as an internal (not necessarily semidirect) product BA, where $B = \langle \beta_1, \beta_2 : \beta_1^{p^{b_1}} = \beta_2^{p^{b_2}} = 1, [\beta_1, \beta_2] = 1\rangle$ and $A = \langle \alpha_1, \alpha_2 : \alpha_1^{p^{a_1}}, \alpha_2^{p^{a_2}}, [\alpha_1, \alpha_2] \in B\rangle$ for some positive integers a_1, a_2, b_1, b_2.

There are two possibilities for $[\alpha_1, \alpha_2]$ that we need to consider separately in order to construct a semidirect product.

(I). If $[\alpha_1, \alpha_2] = 1$ then $\alpha_1^{p^{a_1}}, \alpha_2^{p^{a_2}} \in Z(G) \cap B$. Denote $p^{t_i} = \mathrm{ord}(\alpha_i Z(G))$ for $i = 1, 2$. Define F to be the semidirect product $F = B \rtimes \widetilde{A}$, where $\widetilde{A} = \langle \widetilde{\alpha}_1, \widetilde{\alpha}_2 : \widetilde{\alpha}_1^{p^{t_1}} = \widetilde{\alpha}_2^{p^{t_2}} = 1, [\widetilde{\alpha}_1, \widetilde{\alpha}_2] = 1\rangle$ and $\widetilde{A}$ acts on B in the same way as A on B. Next, define a map $\theta : G/Z(G) \to F/Z(F)$ by $\theta : \alpha_i Z(G) \mapsto \widetilde{\alpha}_i Z(F), \beta_i Z(G) \mapsto \beta_i Z(F)$ for $i = 1, 2$. An easy verification shows that θ is an isomorphism. Since G is of nilpotency class 2, we have $G' \leq Z(G) \cap B$. If $[\beta_i, \alpha_j]$ is a generator of G' (for some i, j), define a map $\phi : G' \to F'$ by $\phi : [\beta_i, \alpha_j] \mapsto [\beta_i, \widetilde{\alpha}_j]$. Therefore, ϕ is an isomorphism and $\widetilde{\alpha}_j Z(F) = \theta(\alpha_j Z(G))$, so G and F are isoclinic.

(II). We will assume henceforth that $[\alpha_1, \alpha_2] \neq 1$. Denote $p^{s_i} = \max\{\mathrm{ord}([\beta_i, \alpha_j]) : 1 \leq j \leq 2\}$ for $i = 1, 2$. It is not hard to see that $\beta_i^{p^{s_i}} \in Z(G) \cap B$ and $\beta_i^{p^k} \notin Z(G) \cap B$ for $k < s_i, i = 1, 2$. If $\beta_1^x \beta_2^y \notin Z(G) \cap B$ for all positive integers $x, y : x < p^{s_1}, y < p^{s_2}$, then $Z(G) \cap B$ is generated by $\beta_1^{p^{s_1}}$ and $\beta_2^{p^{s_2}}$. However, it is possible that $\beta_1^x \beta_2^y \in Z(G) \cap B$ for some $x, y : x < p^{s_1}, y < p^{s_2}$. Choose the minimal integers x and y with this property, and denote $x = x'p^{k_1}, y = y'p^{k_2}$ for some $x', y' : \gcd(x', p) = \gcd(y', p) = 1, 0 \le k_1 < s_1, 0 \le k_2 < s_2$. We can suppose that $k_1 \le k_2$. Then $\beta_1^x \beta_2^y = (\beta_1^{x'} \beta_2^{y'p^{k_2-k_1}})^{p^{k_1}} \in Z(G) \cap B$, and put $\beta = \beta_1^{x'} \beta_2^{y'p^{k_2-k_1}}$. We have now $\beta_1^{x'p^{s_1}} = \beta^{p^{s_1}} \beta_2^{-y'p^{k_2-k_1+s_1}}$, so $\beta_2^{p^{k_2-k_1+s_1}} \in Z(G) \cap B$. Therefore, we must have $k_2 - k_1 + s_1 \ge s_2$. If $k_2 - k_1 + s_1 = s_2$ then $\beta_2^{p^{s_2}} \in \langle \beta_1^{p^{s_1}}, \beta^{p^{k_1}} \rangle$, i.e., $\beta_1^{p^{s_1}}$ and $\beta^{p^{k_1}}$ are the two generators of $Z(G) \cap B$. If $k_2 - k_1 + s_1 > s_2$ then $\beta_1^{p^{s_1}} \in \langle \beta_2^{p^{s_2}}, \beta^{p^{k_1}} \rangle$, i.e., $\beta_2^{p^{s_2}}$ and $\beta^{p^{k_1}}$ are the two generators of $Z(G) \cap B$.

Taking into account what we just observed and the considerations that will follow, we can assume that $Z(G) \cap B$ is generated by some powers of the two generators β_1 and β_2. We are going to consider the following three cases:

I $Z(G) \cap B$ is generated by $\beta_1^{p^{r_1}}$ and $\beta_2^{p^{r_2}}$, where $r_1 \in \mathbb{Z}_{p^{b_1}} \setminus \{0\}, r_2 \in \mathbb{Z}_{p^{b_2}} \setminus \{0\}$.

II $Z(G) \cap B$ is generated by β_1 and $\beta_2^{p^{r_2}}$, where $r_2 \in \mathbb{Z}_{p^{b_2}} \setminus \{0\}$.

III $Z(G) \cap B$ is generated by β_1 and β_2.

Case I. Let $Z(G) \cap B$ be generated by $\beta_1^{p^{r_1}}$ and $\beta_2^{p^{r_2}}$. Denote $c_i = b_i - r_i$ for $i = 1, 2$. Since the commutators of G must lie in the centre of G (recall that G is of nilpotency class 2), we can write $[\beta_i, \alpha_j] = \beta_1^{u_{ij}p^{r_1}} \beta_2^{v_{ij}p^{r_2}}$ for some $u_{ij} \in \mathbb{Z}_{p^{c_1}}, v_{ij} \in \mathbb{Z}_{p^{c_2}}$. We also have $[\alpha_1, \alpha_2] = \beta_1^{ap^{r_1}} \beta_2^{bp^{r_2}}$ for some $a \in \mathbb{Z}_{p^{c_1}}, b \in \mathbb{Z}_{p^{c_2}}$.

Denote

$$\begin{aligned} p^{t_1} &= \max\{\mathrm{ord}([\alpha_1, \alpha_2]), \mathrm{ord}([\beta_1, \alpha_1]), \mathrm{ord}([\beta_2, \alpha_1])\}, \\ p^{t_2} &= \max\{\mathrm{ord}([\alpha_1, \alpha_2]), \mathrm{ord}([\beta_1, \alpha_2]), \mathrm{ord}([\beta_2, \alpha_2])\}. \end{aligned}$$

Hence $\alpha_i^{p^{t_i}} \in Z(G)$ and $\alpha_i^{p^k} \notin Z(G)$ for $k < p^{t_i}$. Let F be a group generated by elements $\widetilde{\alpha}_i, \widetilde{\beta}_i, \gamma_i$ for $i = 1, 2$ such that $\widetilde{\alpha}_i^{p^{t_i}} = \widetilde{\beta}_i^{p^{r_i}} = \gamma_i^{p^{c_i}} =$

$1, [\widetilde{\alpha}_1, \widetilde{\alpha}_2] = \gamma_1^a \gamma_2^b, [\widetilde{\beta}_i, \widetilde{\alpha}_j] = \gamma_1^{u_{ij}} \gamma_2^{v_{ij}}$, where $\gamma_i \in Z(F)$. Without loss of generality, we may assume that $[\widetilde{\alpha}_1, \widetilde{\alpha}_2] = \gamma_1^a$, since we can adjust the central generators γ_1 and γ_2 so that $\gamma_1^a \gamma_2^b$ is a power of one generator. (For example, if $a = a' p^k, b = b' p^l$ and $k \leq l$, we may put $\gamma_1' = \gamma_1^{a'} \gamma_2^{b' p^{l-k}}$, so $[\widetilde{\alpha}_1, \widetilde{\alpha}_2] = \gamma_1'^{p^k}$.)

Define a map $\theta : G/Z(G) \to F/Z(F)$ by $\theta : \alpha_i Z(G) \mapsto \widetilde{\alpha}_i Z(F)$ and $\beta_i Z(G) \mapsto \widetilde{\beta}_i Z(F)$ for $i = 1, 2$. We have $|G/Z(G)| = p^{t_1+t_2+r_1+r_2} = |F/Z(F)|$. Hence θ is an isomorphism. Define a map $\phi : G' \to F'$ by $\phi : \beta_i^{p^{r_i}} \mapsto \gamma_i$ for $i = 1, 2$. Then ϕ is an isomorphism and $\phi([\beta_i, \alpha_j]) = [\widetilde{\beta}_i, \widetilde{\alpha}_j]$, where $\widetilde{\beta}_i Z(F) = \theta(\beta_i Z(G)), \widetilde{\alpha}_j Z(F) = \theta(\alpha_j Z(G))$, so G and F are isoclinic. We are going to show that F is a semidirect product of abelian groups.

We need to simplify the rules of the commutators by suitable replacements of the generators. These replacements can be associated with simultaneous transformations on the rows and columns of the following two matrices:

$$\mathbf{U} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}.$$

Henceforth we are going to apply two types of replacements of the generators that can be illustrated by matrix transformations.

$\mathscr{A}$ Replace the generator $\widetilde{\beta}_2$ with another generator $\beta_2' = \widetilde{\beta}_2 \widetilde{\beta}_1^x$ for some integer x. We have $[\beta_2', \widetilde{\alpha}_j] = \gamma_1^{u_{2j}+xu_{1j}} \gamma_2^{v_{2j}+xv_{1j}}$. In terms of matrices, this replacement corresponds to the simultaneous addition of the first rows of $\mathbf{U}$ and $\mathbf{V}$, multiplied by x, to the second rows. This we can illustrate with the following diagram:

$$\mathbf{U} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \xrightarrow{r_2 + x r_1} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} + x u_{11} & u_{22} + x u_{12} \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \xrightarrow{r_2 + x r_1} \begin{pmatrix} v_{11} & v_{12} \\ v_{21} + x v_{11} & v_{22} + x v_{12} \end{pmatrix}.$$

$\mathscr{B}$ Replace the generator $\widetilde{\alpha}_2$ with another generator $\alpha_2' = \widetilde{\alpha}_2\widetilde{\alpha}_1^y$ for some integer y. We have $[\widetilde{\beta}_i, \alpha_2'] = \gamma_1^{u_{i2}+yu_{i1}}\gamma_2^{v_{i2}+yv_{i1}}$. In terms of matrices, this replacement corresponds to the simultaneous addition of the first columns of $\mathbf{U}$ and $\mathbf{V}$, multiplied by y, to the second columns. This we can illustrate with the following diagram:

$$\mathbf{U} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \xrightarrow{c_2+yc_1} \begin{pmatrix} u_{11} & u_{12}+yu_{11} \\ u_{21} & u_{22}+yu_{21} \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \xrightarrow{c_2+yc_1} \begin{pmatrix} v_{11} & v_{12}+yv_{11} \\ v_{21} & v_{22}+yu_{21} \end{pmatrix}.$$

Next, choose the entry of $\mathbf{V}$ that has the smallest power of p as a divisor. If necessary, we interchange the generators ($\widetilde{\beta}_1 \leftrightarrow \widetilde{\beta}_2, \widetilde{\alpha}_1 \leftrightarrow \widetilde{\alpha}_2$), so that we may assume that this entry is v_{11}. Then there exist integers x, y such that $v_{11}x \equiv -v_{21} \pmod{p^{c_2}}$ and $v_{11}y \equiv -v_{12} \pmod{p^{c_2}}$. We can apply transformations $\mathscr{A}$ and $\mathscr{B}$ to get

$$\mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \xrightarrow[c_2+yc_1]{r_2+xr_1} \begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix}$$

and $\mathbf{U}$ is transformed into a matrix with entries that we will denote again as u_{ij}. We are going to consider the possibilities for the values of u_{12} and u_{21}.

Sub-case I.1. Let $u_{12} \neq 0$ and $\operatorname{ord}(\gamma_1^{u_{12}}) \geq \operatorname{ord}(\gamma_1^a)$. Then $(\gamma_1^{u_{12}})^z = \gamma_1^{-a}$ for some $z \in \mathbb{Z}_{p^{c_1}}$. We can replace $\widetilde{\alpha}_1$ with $\alpha_1' = \widetilde{\alpha}_1\widetilde{\beta}_1^z$, so $[\alpha_1', \widetilde{\alpha}_2] = [\widetilde{\alpha}_1, \widetilde{\alpha}_2][\widetilde{\beta}_1, \widetilde{\alpha}_2]^z = \gamma_1^a\gamma_1^{-a} = 1$. Then F is a semidirect product: $F = \widetilde{B} \rtimes \widetilde{A}$, where $\widetilde{A} = \langle \alpha_1', \widetilde{\alpha}_2 : \alpha_1'^{p^{t_1}} = \widetilde{\alpha}_2^{p^{t_2}} = 1, [\alpha_1', \widetilde{\alpha}_2] = 1\rangle$ and $\widetilde{B} = \langle \widetilde{\beta}_1, \widetilde{\beta}_2, \gamma_1, \gamma_2 : \widetilde{\beta}_1^{p^{r_1}} = \widetilde{\beta}_2^{p^{r_2}} = \gamma_1^{p^{c_1}} = \gamma_2^{p^{c_2}} = 1, [\widetilde{\beta}_1, \widetilde{\beta}_2] = 1\rangle$.

Sub-case I.2. Let $u_{21} \neq 0$ and $\operatorname{ord}(\gamma_1^{u_{21}}) \geq \operatorname{ord}(\gamma_1^a)$. Then $(\gamma_1^{u_{21}})^z = \gamma_1^a$ for some $z \in \mathbb{Z}_{p^{c_1}}$. We can replace $\widetilde{\alpha}_2$ with $\alpha_2' = \widetilde{\alpha}_2\widetilde{\beta}_2^z$, so $[\widetilde{\alpha}_1, \alpha_2'] = [\widetilde{\alpha}_1, \widetilde{\alpha}_2][\widetilde{\beta}_2, \widetilde{\alpha}_1]^{-z} = \gamma_1^a\gamma_1^{-a} = 1$. Then F is a semidirect product: $F =$

$\widetilde{B} \rtimes \widetilde{A}$, where $\widetilde{A} = \langle \widetilde{\alpha}_1, \alpha'_2 : \widetilde{\alpha}_1^{p^{t_1}} = \alpha_2'^{p^{t_2}} = 1, [\widetilde{\alpha}_1, \alpha'_2] = 1\rangle$ and $\widetilde{B} = \langle \widetilde{\beta}_1, \widetilde{\beta}_2, \gamma_1, \gamma_2 : \widetilde{\beta}_1^{p^{r_1}} = \widetilde{\beta}_2^{p^{r_2}} = \gamma_1^{p^{c_1}} = \gamma_2^{p^{c_2}} = 1, [\widetilde{\beta}_1, \widetilde{\beta}_2] = 1\rangle$.

Sub-case I.3. Let $u_{12} \neq 0, u_{21} \neq 0, \mathrm{ord}(\gamma_1^{u_{21}}) < \mathrm{ord}(\gamma_1^a)$ and $\mathrm{ord}(\gamma_1^{u_{12}}) < \mathrm{ord}(\gamma_1^a)$. Then $\gamma_1^{u_{12}} = (\gamma_1^a)^z$ and $\gamma_1^{u_{21}} = (\gamma_1^a)^w$ for some $w, z \in \mathbb{Z}_{p^{c_1}}$. Define $\beta'_1 = \widetilde{\beta}_1 \widetilde{\alpha}_1^{-z}$ and $\beta'_2 = \widetilde{\beta}_2 \widetilde{\alpha}_2^{w}$. We have now $[\beta'_1, \widetilde{\alpha}_2] = [\beta'_2, \widetilde{\alpha}_1] = 1$ and $[\beta'_1, \beta'_2] = \gamma_1^{u_{21}z}$. It is not hard to see that F is a semidirect product: $F = \widetilde{B} \rtimes \widetilde{A}$, where $\widetilde{A} = \langle \widetilde{\beta}_1, \widetilde{\alpha}_2 : \widetilde{\beta}_1^{p^{r_1}} = \widetilde{\alpha}_2^{p^{t_2}} = 1, [\widetilde{\beta}_1, \widetilde{\alpha}_2] = 1\rangle$ and $\widetilde{B} = \langle \widetilde{\beta}_2, \widetilde{\alpha}_1, \gamma_1, \gamma_2 : \widetilde{\beta}_2^{p^{r_2}} = \widetilde{\alpha}_1^{p^{t_1}} = \gamma_1^{p^{c_1}} = \gamma_2^{p^{c_2}} = 1, [\widetilde{\beta}_2, \widetilde{\alpha}_1] = 1\rangle$.

Sub-case I.4. Let $u_{12} = u_{21} = 0$. Then F is a semidirect product: $F = \widetilde{B} \rtimes \widetilde{A}$, where $\widetilde{A} = \langle \widetilde{\beta}_1, \widetilde{\alpha}_2 : \widetilde{\beta}_1^{p^{r_1}} = \widetilde{\alpha}_2^{p^{t_2}} = 1, [\widetilde{\beta}_1, \widetilde{\alpha}_2] = 1\rangle$ and $\widetilde{B} = \langle \widetilde{\beta}_2, \widetilde{\alpha}_1, \gamma_1, \gamma_2 : \widetilde{\beta}_2^{p^{r_2}} = \widetilde{\alpha}_1^{p^{t_1}} = \gamma_1^{p^{c_1}} = \gamma_2^{p^{c_2}} = 1, [\widetilde{\beta}_2, \widetilde{\alpha}_1] = 1\rangle$.

Case II. Let $Z(G) \cap B$ be generated by β_1 and $\beta_2^{p^{r_2}}$. Denote $c_2 = b_2 - r_2$. We can write $[\beta_2, \alpha_j] = \beta_1^{u_{2j}} \beta_2^{v_{2j} p^{r_2}}$ for some $u_{2j} \in \mathbb{Z}_{p^{b_1}}, v_{2j} \in \mathbb{Z}_{p^{c_2}}$. We also have $[\alpha_1, \alpha_2] = \beta_1^a \beta_2^{b p^{r_2}}$ for some $a \in \mathbb{Z}_{p^{b_1}}, b \in \mathbb{Z}_{p^{c_2}}$.

Denote

$$\begin{aligned} p^{t_1} &= \max\{\mathrm{ord}([\alpha_1, \alpha_2]), \mathrm{ord}([\beta_2, \alpha_1])\}, \\ p^{t_2} &= \max\{\mathrm{ord}([\alpha_1, \alpha_2]), \mathrm{ord}([\beta_2, \alpha_2])\}. \end{aligned}$$

Hence $\alpha_i^{p^{t_i}} \in Z(G)$ and $\alpha_i^{p^k} \notin Z(G)$ for $k < p^{t_i}$. Let F be a group generated by elements $\widetilde{\alpha}_i, \gamma_i, i = 1, 2$ and $\widetilde{\beta}_2$ such that $\widetilde{\alpha}_i^{p^{t_i}} = \widetilde{\beta}_2^{p^{r_2}} = \gamma_1^{p^{b_1}} = \gamma_2^{p^{c_2}} = 1, [\widetilde{\alpha}_1, \widetilde{\alpha}_2] = \gamma_1^a \gamma_2^b, [\widetilde{\beta}_2, \widetilde{\alpha}_j] = \gamma_1^{u_{2j}} \gamma_2^{v_{2j}}$, where $\gamma_i \in Z(F)$. Without loss of generality, we may again assume that $[\widetilde{\alpha}_1, \widetilde{\alpha}_2] = \gamma_1^a$.

Define a map $\theta : G/Z(G) \to F/Z(F)$ by $\theta : \alpha_i Z(G) \mapsto \widetilde{\alpha}_i Z(F)$ and $\beta_2 Z(G) \mapsto \widetilde{\beta}_2 Z(F)$ for $i = 1, 2$. We have $|G/Z(G)| = p^{t_1 + t_2 + r_2} = |F/Z(F)|$. Hence θ is an isomorphism. Define a map $\phi : G' \to F'$ by $\phi : \beta_1 \mapsto \gamma_1, \beta_2^{p^{r_2}} \mapsto \gamma_2$. Then ϕ is an isomorphism and $\phi([\beta_2, \alpha_j]) = [\widetilde{\beta}_2, \widetilde{\alpha}_j]$, where $\widetilde{\beta}_2 Z(F) = \theta(\beta_2 Z(G)), \widetilde{\alpha}_j Z(F) = \theta(\alpha_j Z(G))$, so G and F are isoclinic. We are going to show that F is isoclinic to a semidirect product of abelian groups.

Define the matrices

$$\mathbf{U}=\begin{pmatrix}0 & 0\\ u_{21} & u_{22}\end{pmatrix}\quad \text{and}\quad \mathbf{V}=\begin{pmatrix}0 & 0\\ v_{21} & v_{22}\end{pmatrix}.$$

Next, choose the entry of **V** that has the smallest power of p as a divisor. If necessary, we interchange the generators ($\widetilde{\alpha}_1 \leftrightarrow \widetilde{\alpha}_2$), so that we may assume that this entry is v_{22}. Then there exists an integers x such that $v_{22}x \equiv -v_{21} \pmod{p^{c_2}}$. We can apply transformation $\mathscr{B}$ to get

$$\mathbf{V}=\begin{pmatrix}0 & 0\\ v_{21} & v_{22}\end{pmatrix}\xrightarrow[c_1+xc_2]{}\begin{pmatrix}0 & 0\\ 0 & v_{22}\end{pmatrix},$$

and **U** is transformed into a matrix with entries that we will denote again as u_{ij} (of course, we again have $u_{11}=u_{12}=0$). We are going to consider the possibilities for the values of u_{21}.

Sub-case II.1. Let $u_{21}\neq 0$ and $\mathrm{ord}(\gamma_1^{u_{21}})\geq \mathrm{ord}(\gamma_1^a)$. Then $(\gamma_1^{u_{21}})^z=\gamma_1^a$ for some $z\in\mathbb{Z}_{p^{c_1}}$. We can replace $\widetilde{\alpha}_2$ with $\alpha_2'=\widetilde{\alpha}_2\widetilde{\beta}_2^z$, so $[\widetilde{\alpha}_1,\alpha_2']=[\widetilde{\alpha}_1,\widetilde{\alpha}_2][\widetilde{\beta}_2,\widetilde{\alpha}_1]^{-z}=\gamma_1^a\gamma_1^{-a}=1$. Then F is a semidirect product: $F=\widetilde{B}\rtimes\widetilde{A}$, where $\widetilde{A}=\langle\widetilde{\alpha}_1,\alpha_2':\widetilde{\alpha}_1^{p^{t_1}}=\alpha_2'^{p^{t_2}}=1,[\widetilde{\alpha}_1,\alpha_2']=1\rangle$ and $\widetilde{B}=\langle\widetilde{\beta}_2,\gamma_1,\gamma_2:\widetilde{\beta}_2^{p^{r_2}}=\gamma_1^{p^{b_1}}=\gamma_2^{p^{c_2}}=1\rangle$.

Sub-case II.2. Let $u_{21}\neq 0$ and $\mathrm{ord}(\gamma_1^{u_{21}})<\mathrm{ord}(\gamma_1^a)$. Then $\gamma_1^{u_{21}}=(\gamma_1^a)^w$ for some $w\in\mathbb{Z}_{p^{b_1}}$. Define $\beta_2'=\widetilde{\beta}_2\widetilde{\alpha}_2^w$. We have now $[\beta_2',\widetilde{\alpha}_1]=1$. It is not hard to see that F is a semidirect product: $F=\widetilde{B}\rtimes\widetilde{A}$, where $\widetilde{A}=\langle\beta_2',\widetilde{\alpha}_1:\beta_2'^{p^{r_2}}=\widetilde{\alpha}_1^{p^{t_1}}=1,[\beta_2',\widetilde{\alpha}_1]=1\rangle$ and $\widetilde{B}=\langle\widetilde{\alpha}_2,\gamma_1,\gamma_2:\widetilde{\alpha}_2^{p^{t_2}}=\gamma_1^{p^{b_1}}=\gamma_2^{p^{c_2}}=1\rangle$.

Sub-sub-case II.3. Let $u_{21}=0$. Then F is a semidirect product: $F=\widetilde{B}\rtimes\widetilde{A}$, where $\widetilde{A}=\langle\widetilde{\alpha}_1,\widetilde{\beta}_2:\widetilde{\alpha}_1^{p^{t_1}}=\widetilde{\beta}_2^{p^{r_2}}=1,[\widetilde{\beta}_2,\widetilde{\alpha}_1]=1\rangle$ and $\widetilde{B}=\langle\widetilde{\alpha}_2,\gamma_1,\gamma_2:\widetilde{\alpha}_2^{p^{t_2}}=\gamma_1^{p^{b_1}}=\gamma_2^{p^{c_2}}=1\rangle$.

Case III. Let $Z(G)\cap B$ be generated by β_1 and β_2. Then $[\beta_i,\alpha_j]=1$ for all $i,j:1\leq i,j\leq 2$, and $[\alpha_1,\alpha_2]=\beta_1^a\beta_2^b$ for some $a\in\mathbb{Z}_{p^{b_1}}, b\in\mathbb{Z}_{p^{b_2}}$. As we noted before, $\beta_1^a\beta_2^b$ can be written as a power of one generator of B, i.e., we can assume that $[\alpha_1,\alpha_2]=\beta_1^a$ for some $a\in\mathbb{Z}_{p^{b_1}}$. Then

G is an internal product $\widetilde{B}\widetilde{A}$, where $\widetilde{A}$ is an abelian group generated by α_1, β_2, and $\widetilde{B}$ is a normal abelian subgroup of G generated by α_2, β_1. Now we can apply our argument from the beginning of this section to conclude that G is isoclinic to a semidirect product.

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Ivo M. Michailov, Ivan S. Ivanov
Faculty of Mathematics and Informatics, Shumen University,
Universitetska str. 115, 9700 Shumen, Bulgaria
e-mail: ivo_michailov@yahoo.com,
e-mail: slaveicov@abv.bg

Ivaylo A. Dimitrov
Faculty of Mathematics and Informatics, Shumen University,
Universitetska str. 115, 9700 Shumen, Bulgaria
e-mail: pmg.iv.dimitrov@abv.bg