Annual of Konstantin Preslavsky University of Shumen
vol. XX C, 2019, pp. 3 - 11

UNRAMIFIED COHOMOLOGY AND NOETHER'S PROBLEM*

IVO M. MICHAILOV, IVAN S. IVANOV

ABSTRACT: *In this survey we discuss some classical methods and results concerning the present state of Noether's problem and related topics (e.g. Bogomolov multipliers, unramified cohomology).*

KEYWORDS: *Noether's problem, Bogomolov multiplier, unramified cohomology*

2010 Math. Subject Classification: *primary 14E08, 14M20; secondary 13A50,12F12*

1 Introduction

Let K be a field, G a finite group and V a faithful representation of G over K. Then there is a natural action of G upon the field of rational functions $K(V)$.

The rationality problem (also known as *Noether's problem* when G acts on V by permutations) then asks whether the field of G-invariant functions $K(V)^G$ is rational (i.e., purely transcendental) over K.

A question related to the above mentioned is whether $K(V)^G$ is stably rational, that is, whether there exist independent variables $x_1,\dots,x_r$ such that $K(V)^G(x_1,\dots,x_r)$ becomes a purely transcendental extension of K.

This problem has close connection with Lüroth's problem and the inverse Galois problem.

By the noname lemma, if V and V' are two faithful representations of G over K, then $K(V \oplus V')^G$ is rational over both $K(V)^G$ and

*This work is partially supported by a project No RD-08-118/04.02.2019 of Shumen University.

$K(V')^G$. Thus the stable rationality of $K(V)^G$ over K does not depend on the choice of V.

In 1969 and 1972, Swan [13] and Voskresenskiĭ [15] constructed examples for which $\mathbb{Q}(V)^G$ is not rational over $\mathbb{Q}$. (For example if G is a cyclic group of order $47, 113$ and 233.) However their methods do not work over an algebraically closed field of characteristic 0.

In 1984, Saltman [11] gave the first example of a group G such that $\mathbb{C}(V)^G$ is not stably rational over $\mathbb{C}$ using the unramified cohomology group $H^2_{\mathrm{nr}}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ as an obstruction. In a subsequent work Bogomolov [1] made an indepth study of this cohomology group.

More precisely, Bogomolov proved that $H^2_{\mathrm{nr}}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ is canonically isomorphic to

$$B_0(G) = \bigcap_A \ker\{\mathrm{res}_G^A : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})\}$$

where A runs over all the bicyclic subgroups of G (a group A is called bicyclic if A is either a cyclic group or a direct product of two cyclic groups).

Using this isomorphism, he was able to compute explicitly this cohomology group when G is the central extension of an $\mathbb{F}_p$-vector space by another and thus to produce new examples of finite groups G for which $\mathbb{C}(V)^G$ is not stably rational over $\mathbb{C}$.

Let k be a field of characteristic 0, $\overline{k}$ be an algebraic closure of k. For any positive integer n, we denote by μ_n the n-th roots of unity in k and for j in $\mathbb{Z}$ we put

$$\mu_n^{\otimes j} = \begin{cases} \mu_n^{\otimes j-1} \otimes \mu_n & \text{if } j > 1, \\ \mathbb{Z}/n\mathbb{Z}, & \text{if } j = 0, \\ \mathrm{Hom}(\mu_n^{\otimes -j}, \mathbb{Z}/n\mathbb{Z}), & \text{if } j < 0 \end{cases}$$

For $i > 0$, we consider the Galois cohomology groups

$$H^i(k, \mu_n^{\otimes j}) = H^i(\mathrm{Gal}(\overline{k}/k), \mu_n^{\otimes j})$$

as well as their direct limits

$$H^i(k,\mathbb{Q}/\mathbb{Z}(j)) = \varinjlim H^i(k,\mu_n^{\otimes j}).$$

For any function field K over k, we denote by $\mathscr{P}(K/k)$ the set of discrete valuation rings A of rank 1 such that $k \subset A \subset K$ and such that the fraction field $\mathrm{Fr}(A)$ of A is K. If A belongs to $\mathscr{P}(K/k)$, then let κ_A be its residue field and, for any strictly positive integer i and any j in $\mathbb{Z}$,

$$\partial_A : H^i(K,\mu_n^{\otimes j}) \to H^{i-1}(\kappa_A,\mu_n^{\otimes j-1})$$

be the corresponding residue map. They induce residue maps

$$\partial_A : H^i(k,\mathbb{Q}/\mathbb{Z}(j)) \to H^{i-1}(\kappa_A,\mathbb{Q}/\mathbb{Z}(j-1)).$$

Recall that the Brauer group over K is defined by $\mathrm{Br}(K) = H^2(K,K^{s*})$, where K^s is the separable closure of K. $\mathrm{Br}(K)$ is a torsion group and the n-torsion part of the Brauer group is isomorphic to $H^2(K,\mu_n)$.

We have also the residue map

$$\partial_A : \mathrm{Br}(K) \to H^1(\kappa_A,\mathbb{Q}/\mathbb{Z}).$$

Definition 1.1. The unramified cohomology groups are the groups

$$H^i_{\mathrm{nr}}(K,\mu_n^{\otimes j}) = \bigcap_{A\in\mathscr{P}(K/k)} \ker\{H^i(K,\mu_n^{\otimes j}) \xrightarrow{\partial_A} H^{i-1}(\kappa_A,\mu_n^{\otimes j-1})\}.$$

Definition 1.2. The unramified Brauer group is

$$\mathrm{Br}_{\mathrm{nr}}(K) = \bigcap_{A\in\mathscr{P}(K/k)} \ker\{\mathrm{Br}(K) \xrightarrow{\partial_A} H^1(\kappa_A,\mathbb{Q}/\mathbb{Z})\}.$$

Definition 1.3. Here we collect some definitions about fields.

- A field L is a function field over a field K if it is generated by a finite number of elements as a field over K.

- A function field L over K is rational over K if there exist indeterminates $T_1, \ldots, T_m$ and an isomorphism $L \simeq K(T_1, \ldots, T_m)$ over K.

- Two function fields L and M over K are stably isomorphic if there exist indeterminates $U_1, \ldots, U_l, T_1, \ldots, T_m$ and an isomorphism $L(U_1, \ldots, U_l) \simeq M(T_1, \ldots, T_m)$ over K. A function field L is stably rational over K if L is stably isomorphic to K.

- A function field L over K is unirational over K if there exist indeterminates $T_1, \ldots, T_m$ and an injection $L \hookrightarrow K(T_1, \ldots, T_m)$ over K.

Theorem 1.1. *(Colliot-Thélène and Ojanguren [2]) If the function fields K and L are stably isomorphic over k then*

$$H^i_{\mathrm{nr}}(K, \mu_n^{\otimes j}) \simeq H^i_{\mathrm{nr}}(L, \mu_n^{\otimes j}).$$

In particular, if K is stably rational then $H^i_{\mathrm{nr}}(K, \mu_n^{\otimes j}) = \{0\}$.

One can also show that the unramified Brauer group depends only on the stable rationality class of the field. This is the invariant which was used by Artin and Mumford. The unramified cohomology groups may be considered as generalizations of the unramified Brauer group. Indeed, if $i = 2$, the unramified cohomology groups are isomorphic to the n-torsion part of the unramified Brauer group:

$$\mathrm{Br}_{\mathrm{nr}}(K)_{(n)} \simeq H^i_{\mathrm{nr}}(K, \mu_n).$$

We have the following relations between the various kind of rationalities: L rational over $K \Rightarrow L$ stably rational over K; and L stably rational over $K \Rightarrow L$ unirational over K.

Peyre [9] found examples when $H^i_{\mathrm{nr}}(K,\mu_n^{\otimes i}) \neq \{0\}$ for $i = 2,3,4$. The function field K can be taken unirational, but from Colliot-Thélène and Ojanguren's Theorem it follows that K is not stably rational.

In 2008 Peyre [10] also published an example of a group G such that $H^3_{\mathrm{nr}}(\mathbb{C}(V)^G,\mathbb{Q}/\mathbb{Z}) \neq \{0\}$ (and hence $\mathbb{C}(V)^G$ not stably rational over $\mathbb{C}$) although the unramified cohomology group $H^2_{\mathrm{nr}}(\mathbb{C}(V)^G,\mathbb{Q}/\mathbb{Z})$ is trivial.

Kunyavskiĭ called $H^2_{\mathrm{nr}}(\mathbb{C}(V)^G,\mathbb{Q}/\mathbb{Z})$ the Bogomolov multiplier, and we will denote it by $B_0(G)$.

2 Cohomological invariants

Now, let us fix a ground field k_0, and consider the category $\mathrm{Fields}_{/k_0}$ of field extensions k of k_0 and two functors

$$A : \mathrm{Fields}_{/k_0} \longrightarrow \mathrm{Sets} \quad \text{and} \quad H : \mathrm{Fields}_{/k_0} \longrightarrow \text{Abelian Groups}.$$

Definition 2.1. An H-invariant of A is a morphism of functors $a : A \to H$.

Here, we view H as a functor with values in Sets. Hence, $a : A \to H$ means giving, for every $k \in \mathrm{Fields}_{/k_0}$, a map $a_k : E \mapsto a(E)$ of $A(k)$ into $H(k)$ such that, if $\phi : k \to k'$ is a morphism in $\mathrm{Fields}_{/k_0}$, the diagram

$$\begin{array}{ccc} A(k) & \xrightarrow{a_k} & H(k) \\ \big\downarrow & & \big\downarrow \\ A(k') & \xrightarrow{a_{k'}} & H(k') \end{array}$$

is commutative.

Our aim will be to determine explicitly in some cases the group $\mathrm{Inv}(A,H)$ of all such invariants. Consider a finitely generated extension K/k_0; let C be a finite Γ_{k_0}-module.

Definition 2.2. An element $a \in H(K,C)$ is said to be unramified over k_0 if, for every discrete valuation v of K which is trivial on k_0, the residue of a at v is 0.

Definition 2.3. There is a natural embedding $H(k_0) \to \mathrm{Inv}_{k_0}(A,H)$; namely, if $h \in H(k_0)$, we define the invariant a_h by setting $a_h(x) =$ image of h in $H(k)$ for every $x \in A(k)$. Such an invariant is called constant. Suppose we have fixed a base point for A. We say that a is normalized if a vanishes on the base point.

Theorem 2.1. *(Serre [3]) If K/k_0 is rational, every unramified cohomology class in $H(K,C)$ is constant, i.e., belongs to $H(k_0,C)$.*

Theorem 2.2. *(Serre [3]) If $a \in \mathrm{Inv}_{k_0}(G,C)$ is unramified over k_0, and if Noether's problem for G has an affirmative answer over k_0, then a is constant.*

Corollary 2.3. *(Serre [3]) Suppose that Noether's problem for G has an affirmative answer over k_0 and that a is normalized and unramified. Then $a = 0$.*

Serre uses invariants of the trace forms to find these examples.

Corollary 2.4. *(Serre [3]) Let G be a group with a 2-Sylow subgroup which is cyclic of order ≥ 8. Then Noether's problem for G has a negative answer.*

Corollary 2.5. *(Serre [3]) Let G be a group with a 2-Sylow subgroup which is isomorphic to the quaternion group Q_{16} of order 16. Then Noether's problem for G has a negative answer.*

3 The Bogomolov multiplier

We list some of the groups having a trivial Bogomolov multiplier (apart from the groups Noether's problem has an affirmative answer):

(i) abelian extensions of cyclic groups (Bogomolov [1]);

(ii) groups such that the Sylow p-subgroups are cyclic for odd p, and either cyclic, or dihedral, or generalized quaternion for $p = 2$ (Bogomolov [1]);

(iii) simple groups (Kunyavskiĭ [5]);

(iv) Blackburn groups (Kang, Kunyavskiĭ [4]);

(v) unitriangular matrix groups over $\mathbb{F}_p$ and the quotients of their lower central series (Michailov [6]);

(vi) extraspecial p-groups (Kang, Kunyavskiĭ [4]);

(vii) central products of groups with trivial Bogomolov multiplier (open problem posed by Kang and Kunyavskiĭ [4]).

Theorem 3.1. *Let $\theta : G_1 \to G_2$ be a group homomorphism such that its restriction $\theta|_{K_1} : K_1 \to K_2$ is an isomorphism, where $K_1 \leq Z(G_1)$ and $K_2 \leq Z(G_2)$. Let G be a central product of G_1 and G_2, i.e., $G = E/N$, where $E = G_1 \times G_2$ and $N = \{ab : a \in K_1, b \in K_2, \theta(a) = b^{-1}\}$. If $B_0(G_1/K_1) = B_0(G_1) = B_0(G_2) = 0$ then $B_0(G) = 0$.*

Proof. Moravec [8] studied the functor $B_0(G)$, and in particular he found the five term exact sequence

$$B_0(E) \to B_0(E/N) \to \frac{N}{\langle \mathscr{K}(E) \cap N \rangle} \to E^{ab} \to (E/N)^{ab} \to 0,$$

where E is any group, N a normal subgroup of E and $\mathscr{K}(E)$ denotes the set of commutators in E.

If we assume that N is a central subgroup of E, we derive explicitly a three term exact sequence

$$B_0(E) \xrightarrow{\eta_*} B_0(E/N) \xrightarrow{\xi_*} \frac{N \cap [E,E]}{\langle \mathscr{K}(E) \cap N \rangle}. \tag{3.1}$$

We have that $B_0(E) \simeq B_0(G_1) \times B_0(G_2) = 0$. Then from the exact sequence (3.1) it follows that

$$B_0(G) \simeq B_0(G)/\eta_*(B_0(E)) \simeq N_1/N_0,$$

where $N_1 = N \cap [E,E], N_0 = \langle \mathscr{K}(E) \cap N \rangle$. Therefore, we need to show that $N_0 = N_1$, which we achieve by calculations with commutators (for the details see [7, Theorem 3.1]). $\square$

Corollary 3.2. *If G is an extra-special p-group of order p^{2n+1} (for any $n \geq 1$), then $B_0(G) = 0$.*

REFERENCES:

[1] F. A. Bogomolov, The Brauer group of quotient spaces by linear group actions, *Math. USSR Izv.* **30** (1988), 455–485.

[2] J.-L.Colliot-Thélène and M. Ojanguren, Variétés unirationnelles non rationnelles: audelá de l'exemple d'Artin et Mumford, *Invent. Math.* **97** (1989), no. 1, 141–158.

[3] S. Garibaldi, A. Merkurjev and J-P. Serre, "Cohomological invariants in Galois cohomology", *AMS Univ. Lecture Series* vol. **28**, Amer. Math. Soc., Providence, 2003.

[4] Kang M. C., B. Kunyavskiĭ, The Bogomolov multiplier of rigid finite groups, *Arch. Math.* **102,** No. 3 (2014), 209–218.

[5] B. E. Kunyavskiĭ, The Bogomolov multiplier of finite simple groups, *Cohomological and geometric approaches to rationality problems*, 209–217, Progr. Math., **282**, Birkhäuser Boston, Inc., Boston, MA, 2010.

[6] I. Michailov, Bogomolov multipliers for unitriangular groups, *C. R. Acad. Bulg. Sci.* **68** No.6 (2015), 689–696.

[7] I. Michailov, Bogomolov multipliers for some p-groups of nilpotency class 2, *Acta Mathematica Sinica, English Series*, May 2016, Volume **32**, Issue 5, pp 541-552

[8] Moravec P., Unramified Brauer groups of finite and infinite groups, *Amer. J. Math.* **134** (2012),1679–1704.

[9] E. Peyre, Unramified cohomology and rationality problems, *Math. Ann.* **296** (1993), no. 2, 247–268.

[10] E. Peyre, Unramified cohomology in degree 3 and Noether's problem, *Inventiones mathematicae* 2008, Volume **171**, Issue 1, pp 191–225.

[11] D. J. Saltman, Noether's problem over an algebraically closed field, *Invent. Math.* **77** (1984), 71–84.

[12] I. R. Šafarevič, The Lüroth problem, *Proc. Steklov Inst. Math.* **183** (1991), 241-–246.

[13] R. Swan, Invariant rational functions and a problem of Steenrod, *Inventiones Mathematicae*, **7** (2): 148–158, 1969.

[14] R. Swan, Noether's problem in Galois theory, in "Emmy Noether in Bryn Mawr", edited by B. Srinivasan and J. Sally, Springer-Verlag, Berlin, 1983.

[15] V. E. Voskresenskiĭ, Algebraic groups and their birational invariants, *Translations of Mathematical Monographs*, vol. **179**, American Mathematical Society, Providence, RI,1998. Translated from the Russian manuscript by Boris Kunyavski [Boris E. Kunyavskiĭ].

Ivo Michailov
Faculty of Mathematics and Informatics, Shumen University,
Universitetska str. 115, 9700 Shumen, Bulgaria
e-mail: `ivo_michailov@yahoo.com`

Ivan Ivanov
Faculty of Mathematics and Informatics, Shumen University,
Universitetska str. 115, 9700 Shumen, Bulgaria
e-mail: `slaveicov@abv.bg`