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DYNAMICAL BEHAVIOR OF THE STRONG DISPERSIVE NONLINEAR WAVE EQUATION*

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ABSTRACT: *In this paper we consider the dynamical behavior of solutions near explicit self-similar solutions for a strong dispersive nonlinear wave equation. First we construct explicit self-similar solutions, then we investigate dynamical behavior of the solutions near to the self-similar solutions.*

KEYWORDS: *Strong dispersive nonlinear wave equation, Self-similar solutions, Dynamical behavior*

2020 Math. Subject Classification: *35C06*

1 Introduction

In this paper we consider the following strong dispersive nonlinear wave equation

$$u_t - \alpha^2 u_{txx} + 2ku_x + 3uu_x + \gamma(u - \alpha^2 u_{xx})_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}), \tag{1.1}$$

Equation (1.1) is a version of the following well-known generalization of the Dullin-Gottwald-Holm equation [1]

$$u_t - \alpha^2 u_{txx} + 2\omega u_x + 3uu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}), \tag{1.2}$$

Equation (1.2) is derived in [1] as a model for shallow water waves. The Cauchy problem for the equation Dullin-Gottwald-Holm in both periodic and non periodic case was studied in [4, 5, 6]. For (1.2) the problem of the asymptotic stability of self-similar solution was considered in [3]. The authors construct explicit self-similar solutions and consider

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the dynamical behavior of the solutions near to the self-similar solutions. Moreover the asymptotic stability also was considered. Our aim in this paper to construct self-similar solutions of equation (1.1) and to consider dynamical behavior of the solutions arround of these solutions.

Paper is organizated as follows. In section 2 we construct the explicit form the self-similar solutions. In section 3 we investigate dynamical behavior of the solutions near to the self-similar solutions.

2 The explicit self-similar solutions

In this section, we construct self-similar solutions for equation

$$u_t - \alpha^2 u_{txx} + 2ku_x + 3uu_x + \gamma(u - \alpha^2 u_{xx})_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}),$$

$(t,x) \in \mathbb{R}^+ \times \mathbb{R}$.
Let the parameter T be a positive constant. We introduce the similarity coordinates

$$\tau = -log(T-t),\ \rho = \frac{x}{T-t}, \tag{2.1}$$

then we denote by

$$u(t,x) = \phi\left(-log(T-t), \frac{x}{T-t}\right),$$

direct computation gives that
$u_t(t,x) = e^{\tau}(\phi_\tau + \rho\phi_\rho)$, $u_x(t,x) = e^{\tau}\phi_\rho$,
$u_{xx}(t,x) = e^{2\tau}\phi_{\rho\rho}$, $u_{txx} = e^{3\tau}(\phi_{\tau\rho\rho} + 2\phi_{\rho\rho} + \rho\phi_{\rho\rho\rho})$,
$u_{xxx}(t,x) = e^{3\tau}\phi_{\rho\rho\rho}$, $u_{xxxxx}(t,x) = e^{5\tau}\phi_{\rho\rho\rho\rho\rho}$.

Thus Eq. (1.1) is transformed into an one dimensional quasilinear equation

$$\begin{aligned} &\phi_\tau + (\rho + 2k + 3\phi)\phi_\rho - \alpha^2 e^{2\tau}(\phi_{\tau\rho\rho} + 2\phi_{\rho\rho} + \rho\phi_{\rho\rho\rho}) \\ &+\gamma e^{2\tau}(\phi - \alpha^2 e^{2\tau}\phi_{\rho\rho})_{\rho\rho\rho} = \alpha^2 e^{2\tau}(2\phi_\rho\phi_{\rho\rho} + \phi\phi_{\rho\rho\rho}). \end{aligned} \tag{2.2}$$

The steady equation of quasilinear Eq. (2.2) is

$$(\rho+2k+3\phi)\phi_\rho=0, \tag{2.3}$$

which is an ODE. Direct computation shows that it admits a non-trivial solution

$$\phi(\rho)=-\frac{1}{3}(\rho+2k). \tag{2.4}$$

Consequently, the Eq. (1.1) admits an explicit self-similar solution

$$u(t,x)=-\frac{1}{3}\left(\frac{x}{T-t}+2k\right). \tag{2.5}$$

3 Dynamical behavior

Consider the perturbation of the form

$$u(t,x)=v(t,x)+\overline{u}(t,x), \tag{3.1}$$

where $\overline{u}(t,x)=-\frac{1}{3}\left(\frac{x}{T-t}+2k\right)$ is the explicit self-similar solution given in (1.1).

We substitute (3.1) into (1.1), then a dissipative quasilinear equation with singular time coefficient is obtained as

$$\begin{aligned}
&v_t-\alpha^2v_{txx}-\gamma\alpha^2v_{xxxxx}+\left(\gamma+\tfrac{\alpha^2}{3}\left(\tfrac{x}{T-t}+2k\right)\right)v_{xxx}\\
&+\tfrac{2\alpha^2}{3(T-t)}v_{xx}-\tfrac{x}{T-t}v_x-\tfrac{1}{T-t}v\\
&=\alpha^2(2v_xv_{xx}+vv_{xxx})-3vv_x,\ \ \forall(t,x)\in(0,T)\times\mathbb{R},
\end{aligned} \tag{3.2}$$

with the initial data $v(0,x)=v_0(x)=u_0(x)+\frac{1}{3}\left(\frac{x}{T}+2k\right)$.

In the similarity coordinates (2.1), Eq. (3.2) can be rewritten as follows

$$(3.3)\qquad \begin{aligned} &v_\tau-\alpha^2e^{2\tau}v_{\tau\rho\rho}-\tfrac{4\alpha^2}{3}e^{2\tau}v_{\rho\rho}+e^{2\tau}\left(\gamma+\tfrac{2\alpha^2}{3}(k-\rho)\right)v_{\rho\rho\rho}\\ &-\alpha^2\gamma e^{4\tau}v_{\rho\rho\rho\rho\rho}-v+3vv_\rho=\alpha^2e^{2\tau}(2v_\rho v_{\rho\rho}+vv_{\rho\rho\rho}).\end{aligned}$$

We introduce the transformation $\overline{v}(\tau,\rho_0)=e^{-\tau}v(\tau,\rho)$, *with* $\rho_0:=e^{-\tau}\rho$, to reduce Eq. (3.3) into

$$\begin{aligned}&\overline{v}_\tau-\alpha^2\overline{v}_{\tau\rho_0\rho_0}-\frac{\alpha^2}{3}\overline{v}_{\rho_0\rho_0}+e^{-\tau}\left(\gamma(\overline{v}-\alpha^2\overline{v}_{\rho_0\rho_0})_{\rho_0\rho_0\rho_0}+\frac{2\alpha^2}{3}(k-\rho)\overline{v}_{\rho_0\rho_0\rho_0}\right)\\ &+\alpha^2\rho_0\overline{v}_{\rho_0\rho_0\rho_0}+(3\overline{v}-\rho_0)\overline{v}_{\rho_0}=\alpha^2(2\overline{v}_{\rho_0}\overline{v}_{\rho_0\rho_0}+\overline{v}\,\overline{v}_{\rho_0\rho_0\rho_0}).\end{aligned}$$

Note the operator $1-\alpha^2\partial_{\rho_0\rho_0}$ has a fundamental solution $p(x)=\frac{1}{2\alpha}e^{-|\frac{\rho_0}{\alpha}|}$.

We can denote the operator $(1-\alpha^2\partial_{\rho_0\rho_0})^{\frac{1}{2}}$ by Λ, then $\Lambda^{-2}\overline{v}=p(\rho_0)\star\overline{v}$ for all $\overline{v}\in\mathbb{L}^2$.

Let $w(\tau,\rho_0)=\overline{v}(\tau,\rho_0)-\alpha^2\overline{v}_{\rho_0\rho_0}(\tau,\rho_0)$, then it holds $\overline{v}(\tau,\rho_0)=p\star w$, where $\rho_0\in\mathbb{R}$ and $\star$ denotes the convolution. Furthemore, Eq. (3.3) can be rewritten as a dissipative non-local equation

$$(3.4)\qquad \begin{aligned}&w_\tau+\tfrac{1}{3}w-e^{-\tau}\left(\tfrac{2k+e^\tau\rho_0}{3}\right)w_{\rho_0}+\gamma e^{-\tau}w_{\rho_0\rho_0\rho_0}-\tfrac{1}{3}(p\star w)\\ &+e^{-\tau}\left(\tfrac{2(k-e^\tau\rho_0)}{3}\right)(p\star w)_{\rho_0}+3(p\star w)(p\star w)_{\rho_0}\\ &=2(p\star w)_{\rho_0}(p\star w-w)+(p\star w)((p\star w)_{\rho_0}-w_{\rho_0}),\end{aligned}$$

with the initial data

$$(3.5)\qquad \begin{aligned}w(0,\rho_0):=w_0(\rho_0)&=v_0(\rho_0)-\alpha^2v_{\rho_0\rho_0}(0,\rho_0)\\ &=u_0(x)-\alpha^2u_0''(x)+\tfrac{1}{3}\left(\tfrac{x}{T}+2k\right)\end{aligned}$$

and the boundary condition

(3.6)
$$\lim_{|\rho_0|\to+\infty} w(\tau,\rho_0)=0, \quad \lim_{|\rho_0|\to+\infty} w_{\rho_0}(\tau,\rho_0)=0.$$

Here we use $(p\star w)_{\rho_0\rho_0}=\alpha^{-2}(p\star w-w)$.

The term $\left(1-\frac{a}{m}\right)w$ is a dissipative term in Eq. (3.4). This term can make us to get a good priori estimate on the solution for Eq. (3.4). We recall a commutator estimate established in [2].

Lemma 3.1. *[2] Let $s>0$. Then it holds*

(3.7)
$$\|[\Lambda^s,u]v\|_{\mathbb{L}^2}\le C\left(\|\partial_x u\|_{\mathbb{L}^\infty}\|\Lambda^{s-1}v\|_{\mathbb{L}^2}+\|\Lambda^s u\|_{\mathbb{L}^2}\|v\|_{\mathbb{L}^\infty}\right),$$

where positive constant C depending on s.

We now derive a priori estimate of the solution for Eq. (3.4). Let $s>0$. Applying Λ^s to both sides of (3.4), it holds

(3.8)
$$\begin{aligned}&(\Lambda^s w)_\tau+\tfrac{1}{3}\Lambda^s w-e^{-\tau}\Lambda^s\left[\tfrac{2k+e^\tau\rho_0}{3}w_{\rho_0}\right]+\gamma e^{-\tau}\Lambda^s w_{\rho_0\rho_0\rho_0}\\ &-\tfrac{1}{3}\Lambda^s(p\star w)+e^{-\tau}\Lambda^s\left[\tfrac{2(k-e^\tau\rho_0)}{3}(p\star w)_{\rho_0}\right]+3\Lambda^s(p\star w)(p\star w)_{\rho_0}\\ &=2\Lambda^s\left[(p\star w)_{\rho_0}(p\star w-w)\right]+\Lambda^s\left[(p\star w)\left((p\star w)_{\rho_0}-w_{\rho_0}\right)\right].\end{aligned}$$

Lemma 3.2. *Let $s>4$. Then any solution w of Eq. (3.4) satisfies*

$$\| w \|_{\mathbb{H}^s(\mathbb{R})}\le Ce^{-\tau}\| w_0 \|_{\mathbb{H}^s(\mathbb{R})},$$

where C is a positive constant, depending on s.

Proof. Taking the $\mathbb{L}^2$-inner product with equation (3.8) by $\Lambda^s w$, we get

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{d\tau}\|w\|_{\mathbb{H}^s}^2+\tfrac{1}{3}\|w\|_{\mathbb{H}^s}^2-e^{-\tau}\textstyle\int_{\mathbb{R}}\Lambda^s w\Lambda^s\left[\tfrac{2k+e^{\tau}\rho_0}{3}w_{\rho_0}\right]d\rho_0\\
&+\gamma e^{-\tau}\textstyle\int_{\mathbb{R}}\Lambda^s w\Lambda^s[w_{\rho_0\rho_0\rho_0}]d\rho_0-\tfrac{1}{3}\int_{\mathbb{R}}\Lambda^s w\Lambda^s(p\star w)d\rho_0\\
&+e^{-\tau}\textstyle\int_{\mathbb{R}}\Lambda^s w\Lambda^s\left[\tfrac{2(k-e^{\tau}\rho_0)}{3}(p\star w)_{\rho_0}\right]d\rho_0\\
&+3\textstyle\int_{\mathbb{R}}\Lambda^s w\Lambda^s\left((p\star w)(p\star w)_{\rho_0}\right)d\rho_0\\
&=2\textstyle\int_{\mathbb{R}}\Lambda^s w\Lambda^s\left[(p\star w)_{\rho_0}(p\star w-w)\right]d\rho_0\\
&+\textstyle\int_{\mathbb{R}}\Lambda^s w\Lambda^s\left[(p\star w)\left((p\star w)_{\rho_0}-w_{\rho_0}\right)\right]d\rho_0.
\end{aligned}
\tag{3.9}
$$

Next we estimate each of terms in (3.9) . On the hand, we use integration by parts to derive

$$
\begin{aligned}
&\textstyle\int_{\mathbb{R}}\Lambda^s w\Lambda^s\left[\tfrac{2k+e^{\tau}\rho_0}{3}w_{\rho_0}\right]d\rho_0=\int_{\mathbb{R}}\left[\tfrac{2k+e^{\tau}\rho_0}{3}w_{\rho_0}\right]\Lambda^{2s}w d\rho_0\\
&=-\tfrac{1}{3}e^{\tau}\textstyle\int_{\mathbb{R}}\Lambda^s w\Lambda^s w d\rho_0-\tfrac{1}{2}\int_{\mathbb{R}}\tfrac{2k+e^{\tau}\rho_0}{3}(\Lambda^s w)_{\rho_0}^2 d\rho_0\\
&=-\tfrac{1}{6}e^{\tau}\|w\|_{\mathbb{H}^s}^2,
\end{aligned}
\tag{3.10}
$$

and

$$
\int_{\mathbb{R}}\Lambda^s w\Lambda^s\left[w_{\rho_0\rho_0\rho_0}\right]d\rho_0=-\frac{1}{2}\int_{\mathbb{R}}(\Lambda^s w_{\rho_0})_{\rho_0}^2 d\rho_0=0,
\tag{3.11}
$$

and

$$
\tfrac{1}{3}\textstyle\int_{\mathbb{R}}\Lambda^s w\Lambda^s(p\star w)d\rho_0=\tfrac{1}{3}\|w\|_{\mathbb{H}^{s-1}}^2,
\tag{3.12}
$$

and

$$\int_{\mathbb{R}} \Lambda^s w \Lambda^s \left[\frac{2(k-e^\tau\rho_0)}{3}(p\star w)_{\rho_0}\right] d\rho_0$$

$$= \frac{2}{3}e^\tau \int_{\mathbb{R}} \Lambda^{s-1} w \Lambda^{s-1} w d\rho_0 + \frac{1}{2}\int_{\mathbb{R}} \frac{2(k-e^\tau\rho_0)}{3}(\Lambda^{s-1}w)^2_{\rho_0} d\rho_0 \tag{3.13}$$

$$= e^\tau \|w\|^2_{\mathbb{H}^{s-1}},$$

and

$$3\int_{\mathbb{R}} \Lambda^s w \Lambda^s \left((p\star w)(p\star w)_{\rho_0}\right) d\rho_0 = -\frac{3}{2}\int_{\mathbb{R}} w_{\rho_0}(\Lambda^{s-1}w)^2 d\rho_0$$

$$\leq \frac{3}{2} \| w_{\rho_0} \|_{\mathbb{L}^\infty} \| w \|^2_{\mathbb{H}^{s-1}} \leq \frac{3}{2} \| w \|^3_{\mathbb{H}^{s-1}} . \tag{3.14}$$

On the other hand, by (3.7), Holder inequality and $\mathbb{H}^{s-1} \subset \mathbb{L}^\infty$ with $s > 4$, we derive

$$2 \left| \int_{\mathbb{R}} \Lambda^s w \Lambda^s \left((p\star w)_{\rho_0}(p\star w - w)\right) d\rho_0 \right|$$

$$= 2 \left| \int_{\mathbb{R}} [\Lambda^s, (p\star w - w)](p\star w)_{\rho_0} \Lambda^s w d\rho_0 \right|$$
$$+2 \left| \int_{\mathbb{R}} (p\star w - w)\Lambda^s (p\star w)_{\rho_0} \Lambda^s w d\rho_0 \right|$$

$$\leq C\left(\| (p\star w - w)_{\rho_0} \|_{\mathbb{L}^\infty} \| \Lambda^{s-1}(p\star w)_{\rho_0} \|_{\mathbb{L}^2}\right.$$
$$\left. + \| \Lambda^s (p\star w - w) \|_{\mathbb{L}^2} \| (p\star w)_{\rho_0} \|_{\mathbb{L}^\infty}\right) \| w \|_{\mathbb{H}^s} \tag{3.15}$$

$$+2\left(\| p\star w - w \|_{\mathbb{L}^\infty} + \| (p\star w - w)_{\rho_0} \|_{\mathbb{L}^\infty}\right) \| w \|^2_{\mathbb{H}^2}$$

$$\leq C \| w \|^3_{\mathbb{H}^s}$$

and

$$\left|\int_{\mathbb{R}} \Lambda^s w \Lambda^s \left((p\star w)((p\star w)_{\rho_0} - w_{\rho_0})\right) d\rho_0\right|$$

$$= \left|\int_{\mathbb{R}} [\Lambda^s, (p\star w)]((p\star w)_{\rho_0} - w_{\rho_0})\Lambda^s w d\rho_0\right|$$
$$+ \left|\int_{\mathbb{R}} (p\star w)\Lambda^s((p\star w)_{\rho_0} - w_{\rho_0})\Lambda^s w d\rho_0\right|$$

$$\leq C\left(\|(p\star w)_{\rho_0}\|_{\mathbb{L}^\infty} \| \Lambda^{s-1}((p\star w)_{\rho_0} - w_{\rho_0})\|_{\mathbb{L}^2}\right.$$
$$\left. + \|\Lambda^s(p\star w)\|_{\mathbb{L}^2} \| (p\star w)_{\rho_0} - w_{\rho_0} \|_{\mathbb{L}^\infty}\right) \| w \|_{\mathbb{H}^s}$$

$$+2 \| p\star w \|_{\mathbb{L}^\infty} \| w \|^2_{\mathbb{H}^2}$$

$$\leq C\|w\|^3_{\mathbb{H}^s}, \tag{3.16}$$

where C is a positive constant, depending on s.

Thus using (3.11)-(3.16), it follows from (3.9) that

$$\frac{d}{d\tau}\|w\|^2_{\mathbb{H}^s} + \|w\|^2_{\mathbb{H}^s} \leq \frac{d}{d\tau}\|w\|^2_{\mathbb{H}^s} + \|w\|^2_{\mathbb{H}^s} + \frac{4}{3} \| w \|^2_{\mathbb{H}^{s-1}} \leq C \| w \|^3_{\mathbb{H}^s},$$

which is a Bernoulli-type differential inequality, it is equivalent to

$$-\frac{d}{d\tau}\|w\|^{-1}_{\mathbb{H}^s} + \|w\|^{-1}_{\mathbb{H}^s} \leq C,$$

which given that

$$\|w\|_{\mathbb{H}^s} \leq Ce^{-\tau}\|w_0\|_{\mathbb{H}^s}.$$

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