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ON SOME APPLICATIONS OF COUPLED FIXED (OR BEST PROXIMITY) POINTS

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ABSTRACT: *We present some possible generalizations of the notion of coupled fixed (or best proximity) points and their applications. We enrich the class of cyclic maps with coupled fixed (or best) proximity points, so that the technique can be applied not only for symmetric systems. We define a new notion of coupled fixed (or best proximity) points that can be applied in the investigation of existence, uniqueness and stability of market equilibrium in duopoly markets.*

KEYWORDS: *Coupled fixed points, Coupled best proximity points, Modular function space, Partially ordered metric space, Duopoly market*

1 Introduction and Preliminaries

The Banach contraction principle states that in a complete metric space (X,ρ) any contraction map $T : X \to X$ has a fixed point, i.e. $\min\{\rho(x,Tx) : x \in X\} = 0$. A lot of results in modelling real world processes in applied mathematics lead to the problem of finding $\min\{\rho(x,Tx) : x \in X\}$. It may happen that the above minimum is greater than zero. One approach for solving the above mentioned problems uses the notion of a best proximity point is introduced in [7], where a sufficient condition for the existence and the uniqueness of best proximity points in uniformly convex Banach spaces is obtained.

A constructed model may depends on two parameters, i.e. $F : X \times X \to X$. The notion of coupled fixed points [9] ($x = F(x,y)$, $y = F(y,x)$) and of a coupled best proximity points ($\rho(x,F(x,y)) = \rho(y,F(y,x)) = \mathrm{dist}(A,B)$) for an ordered pair (F,G), $F : A\times A \to B$, $G : B\times B \to A$, where $A,B \subset X$ [10, 23], is relevant in this context.

We have tried to present some possible generalizations of the notion of coupled fixed (or best proximity) points, that enrich the set of applications of this notion.

2 Coupled fixed points in partially ordered metric spaces

Following [2, 9] let X be a set and let $\preceq$ be a partial order in X, then $(X,\preceq)$ is called a partially ordered set. We call two elements $x,y \in X$ comparable if either $x \preceq y$ or $y \preceq x$. We denote $x \succeq y$ if $y \preceq x$. We say that $x \prec$ if $x \preceq y$ but $x \neq y$. Let (X,ρ) be a metric space with a partial order $\preceq$, then the triple $(X,\rho,\preceq)$ is called a partially ordered metric space.

Definition 1. *([2, 9]) Let $(X,\preceq)$ be a partially ordered set and let $F : X\times X \to X$. The function F is said to have the mixed monotone property if*

$$\textit{for any } x_1,x_2,y \in X \textit{ such that } x_1 \preceq x_2 \textit{ there holds } F(x_1,y) \preceq F(x_2,y)$$

and

$$\textit{for any } y_1,y_2,x \in X \textit{ such that } y_1 \preceq y_2 \textit{ there holds } F(x,y_1) \succeq F(x,y_2).$$

Definition 2. *([2, 9]) Let $F : X\times X \to X$. An ordered pair $(x,y) \in X\times X$ is called coupled fixed point of F if $x = F(x,y)$ and $y = F(y,x)$.*

Let $(X,\rho,\preceq)$ be a partially ordered complete metric space. We endow the product space $X\times X$ with the following partial order $(u,v) \preceq (x,y)$, provided that $x \succeq u$ and $y \preceq v$ holds simultaneously and with the following metric $d((x,y),(u,v)) = \rho(x,u)+\rho(y,v)$ for $(x,y),(u,v) \in X\times X$.

Every where for a partially ordered metric space $(X,\rho,\preceq)$ we will consider the product space $(X\times X,d,\preceq)$ endowed with the mentioned above partial order and metric.

The first results about existence of coupled fixed points for maps with the mixed monotone property was obtained in [9] and applications was presented for solving the initial value problem for systems of differential equations. Later this results was generalized in [2] and the technique seems to have the potential for further generalizations and applications

Theorem 3. *([2]) Let $F:X\times X\to X$ be a continous mapping having the mixed monotone property on X. Assume that there exists a $k\in[0,1)$ with $\rho(F(x,y),F(u,v)\le\frac{k}{2}(\rho(x,u)+\rho(y,v))$ holds for every $x\ge u$ and every $y\le v$. If there exist $x_0,y_0\in X$ such that $x_0\le F(x_0,y_0)$ and $y_0\ge F(y_0,x_0)$, then F has a coupled fixed point (x,y).*

If in addition every pair of elements has either a lower bound or an uper bound the the coupled fixed point is unique.

Let us mention that Ekeland's variational principle holds for any l.s.c maps $T:X\times X\to\mathbb{R}$, provided that X is a partially ordered complete metric space. Unfortunately, when investigating contraction type of maps $F:X\times X\to X$, satisfying the mixed monotone property in a partially ordered complete metric space $X\times X$, the contraction conditions holds only for part of the points $(x,y),(u,v)\in X\times X$. Thus we can not apply Ekeland's variational principle, as it is done in [6]. A generalization Ekeland's variational principle, on classes of subsets of partially ordered complete metric space $X\times X$, which need not to be compact or even closed, is obtained in [25]. It is applied in the investigations of coupled fixed points for maps that satisfy the mixed monotone property [25].

Let (X,ρ) be a metric space. Following [3] an extended real valued function $T:X\to(-\infty,+\infty]$ on X is called lower semicontinuous (for short l.s.c) if $\{x\in X: f(x)>a\}$ is an open set for each $a\in(-\infty,+\infty]$. Equivalently T is l.s.c if and only if at any point $x_0\in X$ there holds $\liminf\limits_{x\to x_0} f(x)\ge f(x_0)$. A function T is called to be proper function, provided that $T\not\equiv+\infty$.

Theorem 4. *([25]) Let $(X,\rho,\preceq)$ be a partially ordered complete metric space, $(X\times X,d,\preceq)$ and $F:X\times X\to X$ be a continuous map with the mixed monotone property. Let*

$$V\times V=\left\{(x^{(1)},x^{(2)})\in X\times X: x^{(1)}\preceq F\left(x^{(1)},x^{(2)}\right)\ \text{and}\ x^{(2)}\succeq F\left(x^{(2)},x^{(1)}\right)\right\}\ne\emptyset.$$

Let $T:X\times X\to\mathbb{R}\cup\{+\infty\}$ be a proper, l.s.c, bounded from below function. Let $\varepsilon>0$ be arbitrary chosen and fixed and let $u_0\in V\times V$ be an ordered pair such that the inequality $T(u_0)\le\inf_{V\times V}T(v)+\varepsilon$ holds. Then there exists an ordered pair $x\in V\times V$, such that

(i) $T(x)\le\inf\limits_{u\in V\times V}T(u)$;

(ii) $d(x,u_0)\le 1$;

(iii) For every $w\in V\times V$ different from $x\in V\times V$ holds the inequality $T(w)>T(x)-\varepsilon d(w,v)$.

An alternative proof of the next theorem from [1], with the help of Theorem 4 is presented in [25].

Theorem 5. *([25]) Let $(X,\rho,\preceq)$ be a partially ordered complete metric space, $(X\times X,d,\preceq)$ and $F : X\times X \to X$ be a continuous map with the mixed monotone property. Let there exists $\alpha \in [0,1)$, so that the inequality*

$$\rho(F(x,y),F(u,v))+\rho(F(y,x),F(v,u)) \leq \alpha\rho(x,u)+\alpha\rho(y,v) \tag{1}$$

holds for all $x \succeq u$ and $y \preceq v$. If there exists at least one ordered pair (x,y), such that $x \preceq F(x,y)$ and $y \succeq F(y,x)$, then there exists a coupled fixed points (x,y) of F.

If in addition every pair of elements in $X\times X$ has an lower or an upper bound, then the coupled fixed point is unique.

The next results is a corollary of Theorem 5,which slightly generalizes the result from [9].

Corollary 6. *([25]) Let $(X,\rho,\preceq)$ be a partially ordered complete metric space, $(X\times X,d,\preceq)$ and $F : X\times X \to X$ be a continuous map with the mixed monotone property. Let there exists $\alpha,\beta \in [0,1)$, $\alpha+\beta < 1$ so that the inequality*

$$\rho(F(x,y),F(u,v)) \leq \alpha\rho(x,u)+\beta\rho(y,v) \tag{2}$$

holds for all $x \succeq u$ and $y \preceq v$. If there exists at least one ordered pair (x,y), such that $x \preceq F(x,y)$ and $y \succeq F(y,x)$, then there exists a coupled fixed points (x,y) of F.

If in addition every pair of elements in $X\times X$ has an lower or an upper bound, then the coupled fixed point is unique.

Example 7. *([25]) We would like to finish with a particular example. Let $X = \ell_1$, endowed with its classical norm $\|x\|_1 = \sum_{i=1}^{\infty}|x_i|$ and the metric $\rho_1(x,y) = \|x-y\|$. Let us define a partial order in X by $x \preceq y$, if $|x_i| \leq |y_i|$ for all $i \in \mathbb{N}$. Let us define $F : X\times X \to X$ by $F(x,y) = \left\{\frac{|x_i|}{2} - \frac{|y_i|}{3} + \frac{1}{2^i}\right\}_{i=1}^{\infty}$. Let us consider the ordered pair $(x_0,y_0) = \left(\left\{\frac{2}{5.2^i}\right\}_{i=1}^{\infty}, \left\{\frac{2}{2^i}\right\}_{i=1}^{\infty}\right)$. Then the map F satisfies the conditions of Corollary 6 with the additional assumption and consequently F has a unique coupled fixed point.*

It was proved in [4] the existence and uniqueness of coupled fixed points for Kannan type maps in metric space. A generalization in the context of mixed monotone maps in partially ordered metric spaces is resented in [25].

Theorem 8. *([25]) Let $(X,\rho,\preceq)$ be a partially ordered complete metric space, $(X\times X,d,\preceq)$ and $F : X\times X \to X$ be a continuous map with the mixed monotone property. Let there exists $\alpha \in [0,1/2)$, so that the inequality*

$$\rho(F(x,y),F(u,v)) \leq \alpha\rho(x,F(x,y))+\alpha\rho(u,F(u,v)) \tag{3}$$

holds for all $x \succeq u$ and $y \preceq v$. If there exists at least one ordered pair (x,y), such that $x \preceq F(x,y)$ and $y \succeq F(y,x)$, then there exists a coupled fixed point (x,y) of F.

3 Modified Coupled fixed (or best proximity) points

Let (X,ρ) be a metric space. We define a distance between two subsets $A,B \subset X$ by $\mathrm{dist}(A,B) = \inf\{\rho(x,y) : x \in A, y \in B\}$.

Definition 9. *([26]) Let A_x, A_y, B_x and B_y be nonempty subsets of a metric space (X,ρ), $F: A_x\times A_y\to B_x$, $f: A_x\times A_y\to B_y$. An ordered pair $(\xi,\eta)\in A_x\times A_y$ is called a coupled point of (F,f) if*

$$\xi = F(\xi,\eta) \ \text{ and } \ \eta = f(\xi,\eta).$$

Just to fit some of the formulas in the text field let us denote $d_i = d(A_i,B_i)$, $i\in\{x,y\}$.

Definition 10. *([26]) Let A_x, A_y, B_x and B_y be nonempty subsets of a metric space (X,ρ), $F: A_x\times A_y\to B_x$, $f: A_x\times A_y\to B_y$. An ordered pair $(\xi,\eta)\in A_x\times A_y$ is called a coupled best proximity point of (F,f) if*

$$\rho(\xi,F(\xi,\eta)) = \operatorname{dist}(A_x,B_x) \ \text{ and } \ \rho(\eta,f(\xi,\eta)) = \operatorname{dist}(A_y,B_y).$$

Definition 11. *([26]) Let A_x, A_y, B_x and B_y be nonempty subsets of X. Let $F: A_x\times A_y\to B_x$, $f: A_x\times A_y\to B_y$, $G: B_x\times B_y\to A_x$ and $g: B_x\times B_y\to A_y$. For any pair $(x,y)\in A_x\times A_y$ we define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ by $x_0=x$, $y_0=y$ and*

$$\begin{array}{ll} x_{2n+1} = F(x_{2n},y_{2n}), & y_{2n+1} = f(x_{2n},y_{2n}) \\ x_{2n+2} = G(x_{2n+1},y_{2n+1}), & y_{2n+2} = g(x_{2n+1},y_{2n+1}) \end{array}$$

for all $n\geq 0$.

If $f(x,y)=F(y,x)$, $g(x,y)=G(y,x)$, $A_x=A_y=A$ and $B_x=B_y=B$ we get the notions and the results from [1, 2, 9, 10, 13, 14].

3.1 Coupled fixed points

Theorem 12. *([26]) Let A_x, A_y, B_x and B_y be nonempty subsets of a complete metric space (X,ρ), $F: A_x\times A_y\to B_x$, $f: A_x\times A_y\to B_y$, $G: B_x\times B_y\to A_x$ and $g: B_x\times B_y\to A_y$. Let there exist $\alpha,\beta,\gamma,\delta>0$, $\max\{\alpha+\gamma,\beta+\delta\}<1$, such that*

$$\rho(F(x,y),G(u,v))+\rho(f(z,w),g(t,s)) \leq \alpha\rho(x,u)+\beta\rho(y,v)+\gamma\rho(z,t)+\delta\rho(w,s) \tag{4}$$

for all $(x,y)\in A_x\times A_y$, $(u,v)\in B_x\times B_y$, $(z,w)\in A_x\times A_y$ and $(t,s)\in B_x\times B_y$. Then

(I) *There exists a unique pair (ξ,η) in $A\cap B$, which is a common coupled fixed point for the maps F and G. Moreover the iteration sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$, defined in Definition 11 converge to ξ and η respectively.*

(II) *a priori error estimates hold* $\max\{\rho(x_n,\xi),\rho(y_n,\eta)\} \leq \frac{k^n}{1-k}(\rho(x_1,x_0)+\rho(y_1,y_0))$*;*

(III) *a posteriori error estimates hold* $\max\{\rho(x_n,\xi),\rho(y_n,\eta)\} \leq \frac{k}{1-k}(\rho(x_{n-1},x_n)+\rho(y_{n-1},y_n))$*;*

(IV) *The rate of convergence $\rho(x_n,\xi)+\rho(y_n,\eta)\leq k(\rho(x_{n-1},\xi)+(y_{n-1},\eta))$ for the sequences of successive iterations. .*

If put $A_x=A_y=A$, $B_x=B_y=B$, $f(x,y)=F(y,x)$, $g(x,y)=G(y,x)$, $z=y$, $w=x$, $t=v$, $s=u$, $\gamma=\beta$ and $\delta=\alpha$, then we get the results from [14] as corollary. More over it is proven in [26] that in this case $\xi=\eta$.

3.2 Coupled best proximity points

Definition 13. *([26]) Let A_x, A_y, B_x and B_y be nonempty subsets of a metric space (X,ρ), $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. The ordered pair of orderer pairs $((F,f),(G,g))$ is said to be a cyclic contraction ordered pair if there exist non-negative numbers $\alpha,\beta,\gamma,\delta$, such that* $\max\{\alpha+\gamma,\beta+\delta\}<1$ *and there holds the inequality*

$$
\begin{array}{rcl}
S_1 & = & \rho(F(x,y),G(u,v))+\rho(f(z,w),g(t,s)) \\
 & \leq & \alpha\rho(x,u)+\beta\rho(y,v)+\gamma\rho(z,t)+\delta\rho(w,s)+(1-(\alpha+\gamma))d_x+(1-(\beta+\delta))d_y
\end{array}
\tag{5}
$$

for all $(x,y),(z,w)\in A_x\times A_y$ and $(u,v),(t,s)\in B_x\times B_y$.

Just to fit some of the formulas in the text field we will denote $P_{n,m}(x,y)=\|x_n-x_m\|+\|y_n-y_m\|$ and $W_{n,m}(x,y)=P_{n,m}(x,y)-(d_x+d_y)=\|x_n-x_m\|+\|y_n-y_m\|-(d_x+d_y)$, where $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be the sequences defined in Definition 11 and $k=\max\{\alpha+\gamma,\beta+\delta\}$, where $\alpha,\beta,\gamma,\delta$ are the constants from Definition 13.

Theorem 14. *([26]) Let A_x, A_y, B_x and B_y be nonempty convex subsets of a uniformly convex Banach space $(X,\|\cdot\|)$, $F : A_x\times A_y\to B_x$, $f : A_x\times A_y\to B_y$, $G : B_x\times B_y\to A_x$ and $g : B_x\times B_y\to A_y$. Let the ordered pair $((F,f),(G,g))$ be a cyclic contraction. Then (F,f) has a unique coupled best proximity point $(\xi,\eta)\in A_x\times A_y$ and (G,g) has a unique coupled best proximity point $(\zeta,\varsigma)\in B_x\times B_y$, (i.e. $\|\xi-F(\xi,\eta)\|=d_x$, $\|\eta-f(\xi,\eta)\|=d_y$ and $\|\zeta-G(\zeta,\varsigma)\|=d_x$, $\|\varsigma-g(\zeta,\varsigma)\|=d_y$). Moreover $\zeta=F(\xi,\eta)$, $\varsigma=f(\xi,\eta)$, $\xi=G(\zeta,\varsigma)$ and $\eta=g(\zeta,\varsigma)$. For any arbitrary point $(x,y)\in A\times A$ there hold $\lim_{n\to\infty}x_{2n}=\xi$, $\lim_{n\to\infty}y_{2n}=\eta$, $\lim_{n\to\infty}x_{2n+1}=\zeta$, $\lim_{n\to\infty}y_{2n+1}=\varsigma$ and $\|\xi-\zeta\|+\|\eta-\varsigma\|=d_x+d_y$. Moreover there hold*

$$
\begin{array}{c}
G(F(\xi,\eta),f(\xi,\eta))=\xi,\ g(F(\xi,\eta),f(\xi,\eta))=\eta, \\
F(G(\zeta,\varsigma),g(\zeta,\varsigma))=\zeta,\ f(G(\zeta,\varsigma),g(\zeta,\varsigma))=\varsigma.
\end{array}
\tag{6}
$$

If in addition $(X,\|\cdot\|)$ has a modulus of convexity of power type with constants $C>0$ and $q>1$, then

(i) *a priori error estimates hold*

$$
\|\xi-x_{2m}\|\leq P_{0,1}(x,y)\sqrt[q]{\frac{W_{0,1}(x,y)}{Cd_x}}\cdot\frac{\sqrt[q]{k^{2m}}}{1-\sqrt[q]{k^2}};\ \ \|\eta-y_{2m}\|\leq P_{0,1}(x,y)\sqrt[q]{\frac{W_{0,1}(x,y)}{Cd_y}}\cdot\frac{\sqrt[q]{k^{2m}}}{1-\sqrt[q]{k^2}};
$$

(ii) *a posteriori error estimates hold*

$$
\|\xi-x_{2n}\|\leq P_{2n,2n-1}(x,y)\sqrt[q]{\frac{W_{2n,2n-1}(x,y)}{Cd_x}}\cdot\frac{\sqrt[q]{k}}{1-\sqrt[q]{k^2}};
$$

$$
\|\eta-y_{2n}\|\leq P_{2n,2n-1}(x,y)\sqrt[q]{\frac{W_{2n,2n-1}(x,y)}{Cd_y}}\cdot\frac{\sqrt[q]{k}}{1-\sqrt[q]{k^2}}.
$$

If put $A_x=A_y=A$, $B_x=B_y=B$, $f(x,y)=F(y,x)$, $g(x,y)=G(y,x)$, $z=y$, $w=x$, $t=v$, $s=u$, $\gamma=\beta$ and $\delta=\alpha$, then we get the results from [14] as corollary. More over it is proven in [26] that in this case $\xi=\eta$ and $\zeta=\varsigma$.

Let us say that this additional results from [26] ($\xi = \eta$ and $\zeta = \varsigma$ for the case of coupled best proximity points and $\xi = \eta$ for the case of coupled fixed points), is not unnatural, as far as all the applications and/or examples presented in [1, 2, 9, 10, 13, 14] are for symmetric systems of equation.

Example 15. *([26]) Let us consider the system of nonlinear equations:*

$$\left| \begin{array}{rcrcl} 36x & + & e^y & = & e+68 \\ 4\arctan\left(\frac{x}{2}\right) & + & 18y & = & \pi+18. \end{array} \right. \tag{7}$$

Let us consider the functions $F(x,y) = -\frac{x}{8} - \frac{e^y}{32} + \frac{e-60}{32}$, $G(x,y) = -\frac{x}{8} - \frac{e^y}{32} - \frac{e-60}{32}$, $f(x,y) = -\frac{\arctan\left(\frac{x}{2}\right)}{4} - \frac{y}{8} + \frac{\pi-14}{16}$, $g(x,y) = -\frac{\arctan\left(\frac{x}{2}\right)}{4} - \frac{y}{8} - \frac{\pi-14}{16}$. Then $F : [2,+\infty) \times [1,1.5] \to (-\infty,-2]$, $f : [2,+\infty) \times [1,1.5] \to [-1.5,-1]$, $G : (-\infty,-2] \times [-1.5,-1] \to [2,+\infty)$, $g : (-\infty,-2] \times 5 - 1.5,-1] \to [1,1.5]$ and the system

$$\left| \begin{array}{rcl} x - F(x,y) & = & 4 \\ y - f(x,y) & = & 2 \end{array} \right. \tag{8}$$

is equivalent to (7). The ordered pair $((F,f),(G,g))$ is a cyclic contraction with constants $\frac{1}{8}$, $\frac{e^{1.5}}{32}$, $\frac{1}{16}$, $\frac{1}{8}$ therefore and the unique solution of (8) is $(2,1)$.

Some application for solutions of symmetric linear systems is presented in [13, 14] and for arbitrary linear systems [12]. In both articles [12, 13] it is presented a technique of choosing of a parameter μ, so that to increase the rate of convergence up the upper bound of the convergence.

4 Modular function spaces

The class of all nonzero regular convex function modular defined on Ω will be denoted by $\mathfrak{R}$. An extensive presentation of modular function spaces and fixed point theory in modular function spaces can be found in [18, 19, 21, 22].

Definition 16. *([22]) Let ρ be a convex function modular.*

(a) A modular function space is the vector space $L_\rho(\Omega,\Sigma)$, or briefly L_ρ, defined by

$$L_\rho = \{f \in \mathscr{M} : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\};$$

(b) The following formula defines a norm in L_ρ (frequently called Luxemburg norm):

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le 1\right\}.$$

A simple example of a modular function space is L_p endowed with the convex modular $\int_\Omega |f(\omega)|^p d\omega$ and ℓ_p endowed with the convex modular $\sum_{i=1}^p |x_i|^p$.

The notion of uniform convexity is generalized by UCi, $UUCi$ for $i = 1,2$.

Definition 17. *([22]) Let $\rho \in \mathfrak{R}$ and $i \in \{1,2\}$. Let $r > 0$, $\varepsilon > 0$. Define*

$$D_i(r,\varepsilon) = \{(f,g) : f,g \in L_\rho, \rho(f) \le r, \rho(g) \le r, \rho\left(\frac{f-g}{i}\right) \ge \varepsilon r\}.$$

Let

$$\delta_i(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right) : (f,g) \in D_i(r,\varepsilon)\right\} > 0 \ \ \text{if} \ \ D_i(r,\varepsilon) \neq \emptyset$$

and

$$\delta_i(r,\varepsilon) = 1 \ \ \text{if} \ \ D_i(r,\varepsilon) = \emptyset.$$

(i) We say that ρ satisfies (UCi) if for any $r > 0$, $\varepsilon > 0$ there holds the inequality $\delta_i(r,s) > 0$.

(ii) We say that ρ satisfies (UUCi) if for every $s \geq 0$, $\varepsilon > 0$ there exists $\eta_i(s,\varepsilon) > 0$, depending on s and ε such that $\delta_i(r,s) > \eta_i(s,\varepsilon) > 0$ for $r > s$.

Following [23] an iterated sequence for the ordered pair of maps (F,G) is defined.

Definition 18. *([15]) Let A and B be nonempty subsets of a functional modular space X. Let $F : A \times A \to B$ and $G : B \times B \to A$. For any pair $(x,y) \in A \times A$ we define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ by $x_0 = x$, $y_0 = y$ and*

$$\begin{aligned} x_{2n+1} &= F(x_{2n}, y_{2n}), & y_{2n+1} &= F(y_{2n}, x_{2n}) \\ x_{2n+2} &= G(x_{2n+1}, y_{2n+1}), & y_{2n+2} &= G(y_{2n+1}, x_{2n+1}) \end{aligned}$$

for all $n \geq 0$.

4.1 Coupled Fixed Points in Modular Function Spaces

Definition 19. *([15]) Let A and B be nonempty subsets of a functional modular space X, $F : A \times A \to A$. An ordered pair $(x,y) \in A \times A$ is said to be a coupled fixed point of F in A if $x = F(x,y)$ and $y = F(y,x)$.*

Definition 20. *([15]) Let A be nonempty subsets of a functional modular space X, $F : A \times A \to A$ is said to be a ρ–contraction if there exist non-negative numbers α, β, such that $\alpha + \beta < 1$ and there holds the inequality*

$$\rho(F(x,y) - F(u,v)) \leq \alpha\rho(x-u) + \beta\rho(y-v)$$

for all $x,y,u,v \in A$.

Theorem 21. *([15])Let $\rho \in \Re$. Let $A \subset L_\rho$ be nonempty, ρ–closed and ρ–bounded. Let $F : A \times A \to A$ be a ρ–contraction. Then F has unique coupled fixed points $(x,y) \in A$. Moreover for any $(x_0,y_0) \in A$ the sequences $\{x_n\}$, $\{y_n\}$ defined by the equations:*

$$x_1 = F(x_0,y_0), y_1 = F(y_0,x_0), x_{n+1} = F(x_n,y_n), y_{n+1} = F(y_n,x_n) \tag{9}$$

converge to the unique coupled fixed points $(x,y) \in A$ (i.e $\lim_{n\to\infty} \rho(x_n - x) = 0$ and $\lim_{n\to\infty} \rho(y_n - y) = 0$).

4.2 Coupled Best Proximity Points in Modular Function Spaces

Definition 22. *([15]) Let A and B be nonempty subsets of a functional modular space X, $F : A\times A\to B$ and $G : B\times B\to A$. The ordered pair (F,G) is said to be a cyclic contraction if there exist non-negative numbers α,β, such that $\alpha+\beta<1$ and there holds the inequality*

$$\rho(F(x,y)-G(u,v))\le \alpha\rho(x-u)+\beta\rho(y-v)+(1-(\alpha+\beta))d(A,B)$$

for all $(x,y)\in A\times A$ and $(u,v)\in B\times B$.

Definition 23. *([15]) Let A and B be nonempty subsets of a functional modular space X, $F : A\times A\to B$. An ordered pair $(x,y)\in A\times A$ is called a coupled best proximity point of F if*

$$\rho(x-F(x,y))=\rho(y-F(y,x))=d.$$

Theorem 24. *([15]) Let $\rho\in\mathfrak{R}$. Assume that ρ satisfies (UC1), has the Δ_2–property and be uniformly continuous. Let $A,B\subseteq L_\rho$ be ρ–closed, ρ–bounded, convex subsets and $F : A\times A\to B$ and $G : B\times B\to A$ be an order cyclic ρ–contraction pair (F,G). Then there exists a unique order pair $(x,y)\in A\times A$ such that (x,y) is a coupled ρ–best proximity point of F in A (i.e. $\rho(x-F(x,y))+\rho(y-F(y,x))=2d(A,B)$). There holds $x=G(F(x,y),F(y,x))$, $y=G(F(y,x),F(x,y))$ the order pair $(F(y,x),F(x,y))$ is a coupled ρ–best proximity point of G in B. More over for any initial guess $(x_0,y_0)\in A\times A$ the iterated sequences $\{x_n\}$, $\{y_n\}$ defined by*

$$\begin{array}{l} x_{2n+1}=F(x_{2n},y_{2n}),y_{2n+1}=F(y_{2n},x_{2n}),\\ x_{2n+2}=G(x_{2n+1},y_{2n+1}),y_{2n+2}=G(y_{2n+1},x_{2n+1}),\\ n=0,1,2\dots \end{array}\tag{10}$$

satisfied $\lim_{n\to\infty}\rho(x_{2n}-x)=0$, $\lim_{n\to\infty}\rho(y_{2n}-y)=0$, $\lim_{n\to\infty}\rho(x_{2n+1}-F(x,y))=0$, $\lim_{n\to\infty}\rho(y_{2n+1}-F(y,x))=0$.

Definition 25. *([15]) Let A and B be nonempty subsets of a modular function space X, $F : A\times A\to B$ and $G : B\times B\to A$. The ordered pair (F,G) is said to be a cyclic ρ–Kannan contraction pair if there exist non-negative number α, such that $\alpha<1/2$ and there holds the inequality*

$$\rho(F(x,y)-G(u,v))\le\alpha\left(\rho(x-F(x,y))+\rho(u-G(u,v))\right)+(1-2\alpha)d(A,B)$$

for all $(x,y)\in A\times A$ and $(u,v)\in B\times B$.

Theorem 26. *([15]) Let $\rho\in\mathfrak{R}$. Assume that ρ satisfies (UC1), has the Δ_2–property and be uniformly continuous. Let $A,B\subseteq L_\rho$ be ρ–closed, ρ–bounded, convex subsets, $F : A\times A\to B$ and $G : B\times B\to A$ and the ordered pair (F,G) be an cyclic ρ–Kannan contraction pair. Then there exists a unique order pair $(x,y)\in A\times A$ such that (x,y) is a coupled ρ–best proximity points of F in A (i.e. $\rho(x-F(x,y))+\rho(y-F(y,x))=2d(A,B)$). There holds $x=G(F(x,y),F(y,x))$, $y=G(F(y,x),F(x,y))$ the order pair $(F(y,x),F(x,y))$ is a coupled ρ–best proximity points of G in B. More over for any initial guess $(x_0,y_0)\in A\times A$ the iterated sequences $\{x_n\}$, $\{y_n\}$ defined by*

$$\begin{array}{l} x_{2n+1}=F(x_{2n},y_{2n}),\ y_{2n+1}=F(y_{2n},x_{2n}),\\ x_{2n+2}=G(x_{2n+1},y_{2n+1}),y_{2n+2}=G(y_{2n+1},x_{2n+1}),\\ n=0,1,2\dots \end{array}\tag{11}$$

satisfied

$$\lim_{n\to\infty}\rho(x_{2n}-x)=0,\ \lim_{n\to\infty}\rho(y_{2n}-y)=0,\ \lim_{n\to\infty}\rho(x_{2n+1}-F(x,y))=0,\ \lim_{n\to\infty}\rho(y_{2n+1}-F(y,x))=0.$$

4.3 Applications

Example 27. *[15] Let* $p \in [1,+\infty)$*,* $a > 0$*,* $\alpha,\beta \in (0,1)$ *be such that* $\alpha+\beta < 1$ *and* $(1-\alpha-\beta)a = \gamma$*. Let us consider the system of equations*

$$\left| \begin{array}{l} |(1+\alpha)x+\beta y+\gamma|^p = (2a)^p \\ |\alpha x+(1+\beta)y+\gamma|^p = (2a)^p \\ x \geq 0,\ y \geq 0. \end{array} \right. \tag{12}$$

It is easy to check that by using a Computer Algebra Software that the ordered pair (a,a) is a solution of (10) if $p \in \mathbb{N}$. If we try to solve this system for $p \notin \mathbb{N}$ then the computer will give no answer.

Let us consider the space $\mathbb{R}_{|\cdot|^p}$ of all reals endowed with the function modular $\rho_p(\cdot) = |\cdot|^p$, then ρ_p has the Δ_2–property and is uniformly continuous. Thus we can apply Theorem 26 in $\mathbb{R}_p$.

By considering $A = [a,b]$, $B = [-b,-a]$ for $0 < a < b$, $F(x,y) = -\alpha x - \beta y - \gamma$ and $G(x,y) = -\alpha x - \beta y + \gamma$ it follows that (F,G) is an order cyclic ρ–contraction pair and from Theorem 26 and thusthere exists a unique order pair $(x,y) \in A \times A$ such that (x,y) is a coupled ρ–best proximity point of F in A (i.e. $\rho_p(x-F(x,y)) = (2a)^p$ and $\rho_p(y-F(y,x)) = (2a)^p$, which just (10)).

By a suitable choice of α, β, γ, a, b and p we get the examples from [10, 14, 23]

5 A variant of the modified cyclic maps, applied in the investigation of equilibrium in duopoly markets

In order to apply the technique of coupled best proximity points and coupled fixed points in economics we will generalize the mentioned up to now notions.

Let us first start with a duopoly model [8, 24] - two companies competing for same consumers and striving to meet the demand with overall production of $Z = x+y$. The market price is defined as $P(Z) = P(x+y)$, which is the inverse of the demand function. Market players have cost functions $c_1(x)$ and $c_2(y)$, respectively. Assuming that both firms are acting rationally, the profit functions are $\Pi_1(x,y) = xP(x+y) - c_1(x)$ and $\Pi_2(x,y) = yP(x+y) - c_2(y)$ of the first and the second firm, respectively. The goal of each company is to maximize its profit, i.e. $\max\{\Pi_1(x,y) : x,$ assuming that y is fixed$\}$ and $\max\{\Pi_2(x,y) : y,$ assuming that x is fixed$\}$. Provided that functions P and c_i, $i = 1,2$ are differentiable, we get the equations

$$\left| \begin{array}{l} \frac{\partial \Pi_1(x,y)}{\partial x} = P(x+y) + xP'(x+y) - c_1'(x) = 0 \\ \frac{\partial \Pi_2(x,y)}{\partial y} = P(x+y) + yP'(x+y) - c_2'(y) = 0. \end{array} \right. \tag{13}$$

The solution of (13) presents the equilibrium pair of production in the duopoly market [8, 24].

Often equations (13) have solutions in the form of $x = b_1(y)$ and $y = b_2(x)$, which are called response functions [8].

It may turn out difficult or impossible to solve (13) thus it is often advised to search for an approximate solution. Another drawback, when searching of an approximate solution is that it may be not stable. Fortunately we can find an implicit formula for the response function in (13) i.e.

$$x = \frac{c_1'(x) - P(x+y)}{P'(x+y)} = F(x,y) \text{ and } y = \frac{c_2'(y) - P(x+y)}{P'(x+y)} = f(x,y).$$

It is still possible that we may end up with response functions, that do not lead to maximization of the profit Π. As it is often assumed, each participant response depends its own production level and that of the pother payers. E.g. if at a moment n the output quantities are (x_n, y_n), and the first player changes its productions to $x_{n+1} = F(x_n, y_n)$, then the second one will also change its output to $y_{n+1} = f(x_n, y_n)$. We have an equilibrium if there are two productions x and y, such that $x = F(x, y)$ and $y = f(x, y)$, where $F : A_x \times A_y \to A_x$ and $f : A_x \times A_y \to A_y$.

Focusing on response functions, allows to put together Cournot and Bertand models. Indeed let the first company have reaction be $F(X,Y)$ and the second one $f(X,Y)$, where $X = (x, p)$ and $Y = (y, q)$. Here x and y denote the output quantity and (p, q) are the prices set by players. In this case companies can compete in terms of both price and quantity.

Definition 28. *([5]) Let A_x, A_y be nonempty subsets of X. Let $F : A_x \times A_y \to A_x$, $f : A_x \times A_y \to A_y$. For any pair $(x, y) \in A_x \times A_y$ we define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ by $x_0 = x$, $y_0 = y$ and $x_{n+1} = F(x_n, y_n)$, $y_{n+1} = f(x_n, y_n)$ for all $n \geq 0$.*

5.1 Coupled fixed points

Definition 29. *([5]) Let A_x, A_y be nonempty subsets of a metric space (X, ρ), $F : A_x \times A_y \to A_x$, $f : A_x \times A_y \to A_y$. An ordered pair $(\xi, \eta) \in A_x \times A_y$ is called a coupled fixed point of (F, f) if $\xi = F(\xi, \eta)$ and $\eta = f(\xi, \eta)$.*

Definition 30. *([5]) Let A_x, A_y be nonempty subsets of a metric space (X, ρ). Let there exist a subset $D \subseteq A_x \times A_y$ and maps $F : D \to A_x$ and $f : D \to A_y$, such that $(F(x,y), f(x,y)) \subseteq D$ for every $(x, y) \in D$. The ordered pair of ordered pairs (F, f) is said to be a cyclic contraction of type one ordered pair if there exist non-negative numbers α, β, such that $\max\{\alpha + \gamma, \beta + \delta\} < 1$ and there holds the inequality*

$$\rho(F(x,y),F(u,v)) + \rho(f(z,w), f(t,s) \leq \alpha\rho(x,u) + \beta\rho(y,v) + \gamma\rho(z,t) + \delta\rho(w,s) \tag{14}$$

for all $(x,y), (u,v), (z,w), (t,s) \in D$.

Theorem 31. *([5]) Let A_x, A_y be nonempty and closed subsets of a complete metric space (X, ρ). Let there exist a closed subset $D \subseteq A_x \times A_y$ and maps $F : D \to A_x$ and $f : D \to A_y$, such that $(F(x,y), f(x,y)) \subseteq D$ for every $(x, y) \in D$. Let the ordered pair (F, f) be a cyclic contraction of type one. Then*

1. *There exists a unique pair (ξ, η) in D, which is a unique coupled fixed point for the ordered pair (F, f). Moreover the iteration sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$, defined in Definition 38 converge to ξ and η respectively, for any arbitrary chosen initial guess $(x, y) \in A_x \times A_y$;*

2. *a priori error estimates hold* $\max\{\rho(x_n, \xi), \rho(y_n, \eta)\} \leq \frac{k^n}{1-k}(\rho(x_1, x_0) + \rho(y_1, y_0))$;

3. *a posteriori error estimates hold* $\max\{\rho(x_n, \xi), \rho(y_n, \eta)\} \leq \frac{k}{1-k}(\rho(x_{n-1}, x_n) + \rho(y_{n-1}, y_n))$;

4. *rate of convergence for the sequences of successive iterations*

$$\rho(x_n, \xi) + \rho(y_n, \eta) \leq k\left(\rho(x_{n-1}, \xi) + (y_{n-1}, \eta)\right),$$

where $k = \max\{\alpha + \gamma, \beta + \delta\}$.

5.2 Application of coupled fixed points in economics

We can state Theorem 31 in an economic language.

Assumption 32. *([5])Let there is a duopoly market, satisfying the following assumptions:*

1. *The two firms are producing homogeneous goods that are perfect substitutes.*
2. *The first firm can produce qualities from the set A_x and the second firm can produce qualities from the set A_y, where A_x and A_y be closed, nonempty subsets of a complete metric space (X,ρ)*
3. *Let there exist a closed subset $D \subseteq A_x \times A_y$ and maps $F : D \to A_x$ and $f : D \to A_y$, such that $(F(x,y), f(x,y)) \subseteq D$ for every $(x,y) \in D$, be the response functions for firm one and two respectively*
4. *Let there exist $\alpha,\beta,\gamma,\delta > 0$, $\max\{\alpha+\gamma,\beta+\delta\} < 1$, such that the inequality*

$$\rho(F(x,y),F(u,v))+\rho(f(z,w),f(t,s)) \leq \alpha\rho(x,u)+\beta\rho(y,v)+\gamma\rho(z,t)+\delta\rho(w,s) \tag{15}$$

holds for all $(x,y),(u,v),(z,w),(t,s) \in A_x \times A_y$.

We will present an example, where each player is producing a single product, goods being perfect substitutes.

Example 33. *([5])) Let us consider a market with two competing firms, each firm producing just one product, and both goods are perfect substitutes. Let the two firms produce quantities $x \in A_x$ and $y \in A_y$, respectively, where $A_x, A_y \subset [0,+\infty)$ and (X,ρ) be the complete metric space $(\mathbb{R}, |\cdot|)$. Let us consider the response functions of player one be $F(x,y) = 80 - \frac{x}{2} - \frac{y}{8}$, and of player two be $f(x,y) = 70 - \frac{x}{3} - \frac{y}{6}$, $A_x = [0,210]$, $A_y = [0,320]$.*

The ordered pair (F, f) is a cyclic map of type one ant therefore there exist a equilibrium pair (x,y) in the market. We get in this case that the equilibrium pair of the production of the two firms is $(49.51, 45.85)$ and the total production will be $x+y = 95.36$.

Table 1: Values of the iterated sequence (x_n, y_n) if stared with $(40,60)$

n	0	1	2	5	10	20	30
x_n	40	52.5	47.92	49.85	49.49	49.51205	49.51219
y_n	60	46.6	44.72	46.11	45.83	45.85354	45.85366

Table 2: Number n of iterations needed by the a priori estimate if stared with $(100,20)$

ε	0.1	0.01	0.001	0.0001	0.00001
n	41	53	66	79	91

Table 3: Number n of iterations needed by the a posteriori estimate if stared with $(100,20)$

ε	0.1	0.01	0.001	0.0001	0.00001
n	14	18	23	27	32

Let us consider a classical example [24], where the price function is a linear and so are the cost functions of both players.

Example 34. *([5, 24]) Assuming the feasible market price is defined by $P(x,y) = 120 - x - y$, it is expected that additional output x from the first company as well as extra production y of the second one will cause decrease in prices. Therefore under equilibrium conditions $x + y$ will be the total production of the two firms and it will also be reflected in prices. Let the two firms have cost functions equal to $30x$ and $20y$, respectively. The profit of the first one is $\Pi_1(x,y) = xP(x,y) - 30x = x(120 - x - y) - 30x = 90x - x^2 - xy$ and the profit of the second one is $\Pi_2(x,y) = yP(x,y) - 20y = y(120 - x - y) - 20y = 100y - y^2 - xy$. Following Cournot model after solving (13) we get the response functions $F : D \to A_x$ and $f : D \to A_y$ of the two firms $F(y) = \frac{90-y}{2}$ and $f(x) = \frac{100-x}{2}$, where $A_y = [0,90]$, $A_x = [0,100]$ and $D = A_x \times A_y$.*

The ordered pair (F, f) is a cyclic map of type one ant therefore there exist a equilibrium pair (x,y) in the market.

We will present an example, where each player is producing two product types, goods from each type being perfect substitutes.

Let us consider a market with two competing firms, and each firm is producing two product types. Let us assume that each firm produces at least 1 item from each product, i.e. $x = (x_1,x_2), y = (y_1,y_2), x_1,x_2,y_1,y_2 \geq 1$. Let us denote the production of the two players by $x = (x_1,x_2)$ and $y = (y_1,y_2)$, respectively.

Let the market of the two goods be endowed with the p norm.

Example 35. *([5]) Let us consider the response functions $F(x,y) = (F_1(x,y),F_2(x,y))$ and $f(x,y) = (f_1(x,y), f_2(x,y))$ defined by*

$$F(x,y) = \begin{cases} \dfrac{90 - \dfrac{x_1+x_2}{2} - \dfrac{y_1+y_2}{3}}{3}, \\ \dfrac{90 - \dfrac{x_1+x_2}{2} - \dfrac{y_1+y_2}{3}}{3}; \end{cases} \quad f(x,y) = \begin{cases} \dfrac{100 - \dfrac{x_1+x_2}{4} - \dfrac{y_1+y_2}{3}}{4} \\ \dfrac{100 - \dfrac{x_1+x_2}{4} - \dfrac{y_1+y_2}{3}}{4}. \end{cases}$$

where

1. $A_x = [0,30] \times [0,30]$ *and* $A_y = [0,25] \times [0,25]$
2. $D = [0,30] \times [0,30] \times [0,25] \times [0,25]$

The ordered pair (F, f) satisfies Assumption 32 with constants $\alpha = \frac{2^{\frac{p-1}{p}}}{3}$, $\beta = \frac{2^{\frac{p-1}{p}+1}}{9}$, $\gamma = \frac{2^{\frac{p-1}{p}}}{6}$ and $\delta = \frac{2^{\frac{p-1}{p}+1}}{9}$. Thus there exists an equilibrium pair (x,y).

We will present an example, where the players are producing a single product and compete on both quantities and prices.

There is a large number of goods where companies can compete on both quality and prices. In this case the equilibrium would depend on balanced decision on what market share to target at a reasonable price. Lets assume that there are only two major players that produce homogeneous products. The first company can produce qualities from the set $A_x \subseteq [0,\infty)$ at a price $p \in P_x \subseteq [0,\infty)$ and the second one can produce qualities from the set $A_y \subseteq [0,\infty)$ at a price $p \in P_x \subseteq [0,\infty)$, where A_x, A_y, P_x, P_y be nonempty subsets. Let $A_x \times P_x$, $A_y \times P_y$ be subsets of a complete metric space $(\mathbb{R}^2, \rho)$.

Assumption 36. *([5])Let there is a duopoly market, satisfying the following assumptions:*

1. *The two firms are producing homogeneous, perfect substitute products.*
2. *The first firm can produce qualities from the set A_x at a price $p \in P_x$ and the second firm can produce qualities from the set A_y at a price $p \in P_x$, where $A_x \times P_x$, $A_y \times P_y$ be nonempty, closed subsets of a complete metric space $(\mathbb{R}^2, \rho)$.*
3. *Let there exists a closed subset $D \subseteq A_x \times P_x \times A_y \times P_y \to A_x$, such that $F : D \to A_x \times P_x$, $f : D \to A_y \times P_y$ and $(F(x,p,y,q), f(x,p,y,q)) \subseteq D$ for every $(x,p,y,q) \in D$ be the response functions for firm one and two respectively.*
4. *Let there exist $\alpha, \beta, \gamma, \delta > 0$, $\max\{\alpha+\gamma, \beta+\delta\} < 1$, such that the inequality*
(16)
$$\rho(F(X,Y),F(U,V)) + \rho(f(Z,W), f(T,S)) \leq \alpha\rho(X,U) + \beta\rho(Y,V) + \gamma\rho(Z,T) + \delta\rho(W,S),$$
where we use thenotations $X = (x, p_1)$, $Y = (y, q_1)$, $U = (u, p_2)$, $V = (v, q2)$, $Z = (z, p_3)$, $W = (w, q_3)$, $T = (t, p_4)$, $S = (s, q_4)$, holds for all $(x, p_1, y, q_1), (u, p_2, v, q_2), (z, p_3, w, q_3)$, $(t, p_4, s, q_4) \in D$.

Then there exists a unique pair (ξ, p, η, q) in $A_x \times P_x \times A_y \times P_y$, which is a common coupled fixed point for the maps F and f, i.e. a market equilibrium pair. Moreover the iteration sequences $\{x_n\}_{n=0}^{\infty}$, $\{p_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$, defined in Definition 11 converge to ξ, p, η, and q respectively and the error estimates from Theorem 31 hold.

Example 37. *([5]) Let us consider a market with two competing firms, producing the same product, and selling it at a price p and q respectively, i.e. $X = (x, p), Y = (y, q)$. Let us consider the response functions $F(X,Y) = (F_1(X,Y), F_2(X,Y))$ and $f(X,Y) = (f_1(X,Y), f_2(X,Y))$ defined by*

$$F(X,Y) = \begin{cases} \dfrac{90 - \frac{x}{2} - \frac{y}{3}}{3}, \\ \dfrac{4 - \frac{p}{2} - \frac{q}{3}}{3}; \end{cases} \quad f(X,Y) = \begin{cases} \dfrac{100 - \frac{x}{4} - \frac{y}{3}}{4} \\ \dfrac{5 - \frac{p}{4} - \frac{q}{3}}{4}. \end{cases}$$

Let $X = (x, p)$ and $Y = (y, q)$ be subsets of $(\mathbb{R}^2, \|\cdot\|_2)$ (the two dimensional Euclidean space). Let

1. *$A_x = [0,100] \times [0,5]$ and $A_y = [0,100] \times [0,4]$*
2. *$D = [0,100] \times [0,5] \times [0,100] \times [0,4]$*

The ordered pair (F, f) satisfies Assumption 36 with constants $\alpha = \frac{1}{3\sqrt{2}}$, $\beta = \frac{1}{3\sqrt{2}}$, $\gamma = \frac{1}{4\sqrt{2}}$ and $\delta = \frac{1}{4\sqrt{2}}$. Thus there exists a unique equilibrium pair (x, y).

5.3 A variational technique in the investigation of equilibrium in duopoly markets

We modify Definition 1 in order to apply a result about coupled fixed points in partially ordered metric spaces with functions that satisfy the mixed monotone property.

Definition 38. *([17]) Let $(Z,\preceq)$ be a partially ordered and $X,Y \subseteq Z$, let $F : X \times Y \to X$ and $f : X \times Y \to Y$. The the ordered couple (F,f) is said to have the mixed monotone property if*

$$\text{for any } x_1,x_2,y \in X \text{ such that } x_1 \preceq x_2 \text{ there holds } F(x_1,y) \preceq F(x_2,y)$$

and

$$\text{for any } y_1,y_2,x \in X \text{ such that } y_1 \preceq y_2 \text{ there holds } f(x,y_1) \succeq f(x,y_2).$$

By using a variation of Theorem 4 we get the next result.

Theorem 39. *([17]) Let there is an oligopoly and let us assume that the two firms are producing homogeneous product completely replaceable with each other. The first firm can produce qualities from the set X and the second firm can produce qualities from the set Y, where X and Y be nonempty subsets of a partially ordered complete metric space $(Z,\rho,\preceq)$. Let $F : X \times Y \to X$, $f : X \times Y \to Y$ be the response functions for firm one and two respectively. Let there exists $\alpha \in (0,1)$, such that*

$$\rho(F(x,y),F(u,v)) + \rho(f(x,y),f(u,v)) \leq \alpha\rho(x,u) + \alpha\rho(y,v) \tag{17}$$

holds for all $x \succeq u$ and $y \preceq v$. If there exists at least one ordered pair $(x,y) \in X \times Y$, such that $x \preceq F(x,y)$ and $y \succeq f(x,y)$, then there exists a market equilibrium point (x,y), which is a coupled fixed points of (F,f).

If in addition every pair of elements in $X \times Y$ has an lower or an upper bound, then the coupled fixed point is unique.

Example 40. *([17]) Let us consider a market with two competing firms, each firm produces two product and any one of the items is completely replaceable with the similar product of the other firm. Let us assume that the the second firm enters the market, i.e. if the productions are (x_1,x_2) and (y_1,y_2) of the first and the second firm, respectively, then $(x_1,x_2) \succeq (y_1,y_2)$. Let endow the production set $\mathbb{R}$ with the euclidean norm $\|\cdot\|_2$. Let us consider the response functions $F(x_1,x_2,y_1,y_2)$ and $f(x_1,x_2,y_1,y_2)$ defined by*

$$F(x,y) = \left\{ \begin{array}{l} \frac{x_1+y_1}{3}+1 \\ \frac{x_2+y_2}{4}+1 \end{array} \right. , \quad f(x,y) = \left\{ \begin{array}{l} \frac{x_1+y_1}{3}+1 \\ \frac{x_2+y_2}{2}+1 \end{array} \right. .$$

The ordered pair (F,f) satisfies Theorem 39 and therefor there exists a market equilibrium.

the partial order and the assumptions may be considered as a second player enters the economy. In this case the first player will decrease its production and the second one will increase it.

5.4 Coupled best proximity points

Definition 41. *([5]) Let A_x, A_y be nonempty subsets of a metric space (X,ρ), $F : A_x \times A_y \to A_x$, $f : A_x \times A_y \to A_y$. An ordered pair $(\xi,\eta) \in A_x \times A_y$ is called a coupled best proximity point of (F,f) if $\rho(\eta,F(\xi,\eta) = \rho(\xi,f(\xi,\eta) = \mathrm{dist}(A_x,A_y)$.*

Definition 42. *([5]) Let A_x, A_y be nonempty subsets of a metric space (X,ρ). Let there exist a subset $D \subseteq A_x \times A_y$ and maps $F : D \to A_x$ and $f : D \to A_y$, such that $(F(x,y),f(x,y)) \subseteq D$ for every $(x,y) \in D$. The ordered pair of ordered pairs (F,f) is said to be a cyclic contraction of type two* *ordered pair if there exist non-negative numbers α,β, such that $\alpha+\beta<1$ and there holds the inequality*

$$\rho(F(x,y),f(u,v))\leq\alpha\rho(x,v)+\beta\rho(y,u)+(1-(\alpha+\beta))\mathrm{dist}(A_x,A_y) \tag{18}$$

for all $(x,y),(u,v)\in D$.

Simply to fit a few of the equations within the content field let us denote $d=\mathrm{dist}(A_x,A_y)$, $P_{n,m}(x,y)=\|x_n-y_m\|$ and $W_{n,m}(x,y)=P_{n,m}(x,y)-d=\|x_n-y_m\|-d$, where $x=\{x_n\}_{n=0}^{\infty}$ and $y=\{y_n\}_{n=0}^{\infty}$.

Theorem 43. *([5]) Let A_x, A_y be nonempty, closed and convex subsets of a uniformly convex Banach space $(X,\|\cdot\|)$. Let there exist a closed and convex subset $D\subseteq A_x\times A_y$ and maps $F:D\to A_x$ and $f:D\to A_y$, such that $(F(x,y),f(x,y)\subseteq D$ for every $(x,y)\in D$. Let the ordered pair (F,f) be a cyclic contraction of type two. Then (F,f) has a unique coupled best proximity point $(\xi,\eta)\in A_x\times A_y$, (i.e. $\|\eta-F(\xi,\eta)\|=\|\xi-f(\xi,\eta)\|=d$). For any initial guess $(x,y)\in A_x\times A_y$ there hold $\lim_{n\to\infty}x_n=\xi$, $\lim_{n\to\infty}y_n=\eta$, $\|\xi-\eta\|=d$, $\xi=F(\xi,\eta)$ and $\eta=f(\xi,\eta)$.*

If in addition $(X,\|\cdot\|)$ has a modulus of convexity of power type with constants $C>0$ and $q>1$, then

1. *a priori error estimates hold*

$$\|\xi-x_m\|\leq M_0\sqrt[q]{\frac{\max\{W_{0,1}(x,y),W_{0,0}(x,y)\}}{Cd}}\cdot\frac{\sqrt[q]{(\alpha+\beta)^m}}{1-\sqrt[q]{\alpha+\beta}};$$

$$\|\eta-y_m\|\leq N_0\sqrt[q]{\frac{\max\{W_{0,1}(y,x),W_{0,0}(y,x)\}}{Cd}}\cdot\frac{\sqrt[q]{(\alpha+\beta)^m}}{1-\sqrt[q]{\alpha+\beta}};$$

2. *a posteriori error estimates hold*

$$\|\xi-x_n\|\leq M_{n-1}\sqrt[q]{\frac{\max\{W_{n-1,n}(x,y),W_{n-1,n-1}(x,y)\}}{Cd}}c;$$

$$\|\eta-y_n\|\leq N_{n-1}\sqrt[q]{\frac{\max\{W_{n-1,n}(y,x),W_{n-1,n-1}(y,x)\}}{Cd}}c,$$

where $M_n=\max\{\|x_n-y_n\|,\|x_n-y_{n+1}\|\}$, $N_n=\max\{\|x_n-y_n\|,\|y_n-x_{n+1}\|\}$ and $c=\frac{\sqrt[q]{\alpha+\beta}}{1-\sqrt[q]{\alpha+\beta}}$.

5.5 Players' production sets have an empty intersection

Theorem 43 can be stated in the economy language.

Assumption 44. *([5])Let there is a duopoly market, satisfying the following assumptions:*

1. *The two firms are producing homogeneous perfect substitute products.*
2. *The first firm can produce qualities from the set A_x and the second firm can produce qualities from the set A_y, where A_x and A_y be nonempty, closed and convex subsets of a uniformly convex Banach space $(X,\|\cdot\|)$*

3. *Let there exist a closed and convex subset $D \subseteq A_x \times A_y$ and maps $F : D \to A_x$ and $f : D \to A_y$, such that $(F(x,y), f(x,y)) \subseteq D$ for every $(x,y) \in D$, be the response functions for firm one and two respectively*

4. *Let there exist $\alpha, \beta > 0$, $\alpha + \beta < 1$, such that*

$$\|F(x,y) - f(u,v)\| \leq \alpha\|x - v\| + \beta\|y - u\| + (1 - (\alpha + \beta))d \tag{19}$$

for all $(x,y), (u,v) \in A_x \times A_y$, where $d = \operatorname{dist}(A_x, A_y) = \inf\{\|x - y\| : x \in A_x, y \in A_y\}$.

Then there exists a unique pair (ξ, η) in $A_x \times A_y$, which is a coupled best point for the pair of maps (F, f), i.e. a market equilibrium pair. Moreover the iteration sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$, defined in Definition 11 converge to ξ and η respectively.

If in addition $(X, \|\cdot\|)$ has a modulus of convexity of power type with constants $C > 0$ and $q > 1$, then the error estimates from Theorem 43 hold.

We will illustrate last result with an example, where players' production sets have an empty intersection, each player is producing two goods.

Example 45. *([5]) Let us consider a market with two competing firms, each firm produces two product and any one of the items is completely replaceable with the similar product of the other firm. Let us assume that the first firm can produce much less quantities than the second one, i.e. if x_1, x_2 be the quantities produced by the first firm and y_1, y_2 be the quantities produced by the second one and, then $x_1, x_2 \in [0,1]$ and $y_1, y_2 \in [2,3]$. Let $A_x = [0,1] \times [0,1]$ $A_y = [2,3] \times [2,3]$ be considered as subsets of $(\mathbb{R}^2, \|\cdot\|_2)$. Let us consider the response functions $F(x_1,x_2,y_1,y_2)$ and $f(x_1,x_2,y_1,y_2)$ defined by*

$$F(x,y) = \begin{cases} \frac{3x_1}{8} + \frac{x_2}{8} - \frac{3y_1}{16} - \frac{y_2}{16} + 1 \\ \frac{x_1}{8} + \frac{3x_2}{8} - \frac{y_1}{16} - \frac{3y_2}{16} + 1 \end{cases}, \quad f(x,y) = \begin{cases} -\frac{3x_1}{16} - \frac{x_2}{16} + \frac{3y_1}{4} + \frac{y_2}{4} + \frac{5}{4} \\ -\frac{x_1}{16} - \frac{3x_2}{16} + \frac{y_1}{4} + \frac{3y_2}{4} + \frac{5}{4} \end{cases}.$$

The ordered pair (F, f) satisfies Assumption 44 with constants $\alpha = \frac{\sqrt{3}}{4}$, $\beta = \frac{\sqrt{3}}{8}$. Thus there exists an equilibrium pair $(x,y) = ((x_1,x_2),(y_1,y_2))$.

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