A GENERALIZED FORM OF THE GELFAND - LEVITAN - MARCHENKO EQUATION

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ABSTRACT: *In this paper we derive a generalized form of the Gelfand-Levitan-Marchenko equation of the inverse scattering problem for the Korteweg-de Vries equation using the obtained output realization of two operator colligation and results of the author concerning the connection between the soliton theory and the theory of commuting nonselfadjoint operators. The presented results can be expanded for the Schrödinger equation, the Heisenberg equation, the Sine-Gordon equation and the Davey-Stewartson equation.*

KEYWORDS: *Nonselfadjoint operator, dissipative operator, operator colligation, triangular model, coupling, Gelfand-Levitan-Marchenko equation*

This paper is dedicated to applications of the connection between the soliton theory and the theory of commuting nonselfadjoint operators. This connection is established by M.S. Livšic and Y. Avishai in [9] and it is based on the Marchenko method for solving of nonlinear differential equations [12] and the Livšic colligation theory of nonselfadjoint operators [10, 11]. M.S. Livšic and Y. Avishai in [9] consider the imaginary part of the dissipative operator B in a Hilbert space H with zero limit

$$\lim_{x\to+\infty}(e^{ixB}f, e^{ixB}f) = 0 \quad (f \in H)$$

as one dimensional subspace of H according to a new scalar product and this allows to apply the connection between theories for obtaining of new scalar solutions of nonlinear differential equations. It turns out that this idea can be expanded in the case of the larger class of nondissipative nonselfadjoint operators B, presented as a coupling of a dissipative operator and an antidissipative operator with absolutely continuous real spectra and nonzero limit

$$\lim_{x\to+\infty}(e^{ixB}f, e^{ixB}f) \neq 0 \quad (f \in H)$$

(see [3]). With the help of the triangular model of the coupling B, introduced in [1], and asymptotics of the corresponding nondissipative curves $\{e^{ixB}f\}$ as $x \to \pm\infty$ (see [5]) scalar solutions of different nonlinear differential equations (the Schrödinger equation, the Heisenberg equation, the Sine-Gordon equation, the Davey-Stewartson equation) are obtained in [3], using appropriate couples and triplets of nonselfadjoint operators.

In this paper we derive a generalized form of the Gelfand-Levitan-Marchenko equation of the inverse scattering problem for the Korteweg-de Vries equation, using results, obtained in [3] and the so-called output realization of two-operator colligation.

Let A and B be linear bounded nonselfadjoint operators in a Hilbert space H with finite dimensional imaginary parts. The subspace $G = G_A + G_B$, where $G_A = (A - A)^*H$ and $G_B = (B - B^*)H$, is called the non-Hermitian subspace of the pair (A, B), G_A, G_B are called non-Hermitian subspaces of A and B correspondingly.

Let H, E be Hilbert spaces, let A, B be linear bounded nonselfadjoint operators in H. The set

$$X = (A, B; H, \Phi, E; \sigma_A, \sigma_B, \gamma, \widetilde{\gamma}) \tag{1}$$

is called a regular colligation, if

$$\frac{1}{i}(A - A^*) = \Phi^*\sigma_A\Phi, \qquad \frac{1}{i}(B - B^*) = \Phi^*\sigma_B\Phi, \tag{2}$$

$$\sigma_A\Phi B^* - \sigma_B\Phi A^* = \gamma\Phi, \tag{3}$$

$$\sigma_A\Phi B - \sigma_B\Phi A = \widetilde{\gamma}\Phi, \tag{4}$$

$$\widetilde{\gamma} - \gamma = i(\sigma_A\Phi\Phi^*\sigma_B - \sigma_B\Phi\Phi^*\sigma_A), \tag{5}$$

where σ_A, σ_B, γ, $\widetilde{\gamma}$ are bounded selfadjoint operators in E, Φ is a bounded linear mapping of H into E.

Instead of the term *regular colligation* it can be used the term *vessel*, that has been coined in [11].

If $\Phi H = E$ and $\ker\sigma_A \cap \ker\sigma_B = \{0\}$ the colligation is called a strict colligation. A colligation is said to be commutative if $AB = BA$. Strict commutative colligations are regular (see [7, 8]).

In many cases of interests the subspace G is finite dimensional. We consider the case when $\dim E < +\infty$.

It has to mention that every pair (A, B) of commuting nonselfadjoint operators in H with finite dimensional imaginary parts can be always embedded in a commutative regular colligation (1) by setting

$$E = G = (A - A^*)H + (B - B^*)H = G_A + G_B, \quad \Phi = P_E,$$

$$\sigma_A = \frac{1}{i}(A - A^*)|_E, \quad \sigma_B = \frac{1}{i}(B - B^*)|_E,$$

$$\gamma = \frac{1}{i}(AB^* - BA^*)|_E, \quad \widetilde{\gamma} = \frac{1}{i}(B^*A - A^*B)|_E$$

where P_E is an orthoprojector onto E.

The system-theoretic interpretation of an two-operator colligation leads to an open two dimensional system. To a given commutative regular colligation (1) there corresponds the next open system

$$\begin{cases} i\frac{\partial f}{\partial t} + Af = \Phi^*\sigma_A u \\ i\frac{\partial f}{\partial x} + Bf = \Phi^*\sigma_B u \\ v = u - i\Phi f \end{cases} \tag{6}$$

where the functions $u = u(x,t)$, $v = v(x,t)$, with values in E and $f = f(x,t)$ with values in H are the collective input, the collective output and the collective state correspondingly.

Instead of the open system (6) we consider a generalized open system, introduced in [3], from the form

$$\begin{cases} i\frac{1}{\varepsilon}\frac{\partial}{\partial t} + Af = \Phi^*\sigma_A u, \\ i\frac{1}{\delta}\frac{\partial}{\partial x}f + Bf = \Phi^*\sigma_B u, \\ v = u - i\Phi f, \end{cases} \tag{7}$$

where constants ε, $\delta \in \mathbb{C}$. Using the equality of the mixed partials $\frac{\partial^2 f}{\partial t\partial x} = \frac{\partial^2 f}{\partial x\partial t}$ it follows the next compatibility conditions:

Theorem 1. *(See [3].) For the commutative regular colligation (1) the equations (7) of the collective motions are compatible if and only if the input $u(x,t)$ satisfies the following partial differential equation*

$$\sigma_B\left(-i\frac{1}{\varepsilon}\frac{\partial u}{\partial t}\right) - \sigma_A\left(-i\frac{1}{\delta}\frac{\partial u}{\partial x}\right) + \gamma u = 0 \tag{8}$$

and the corresponding output $v(x,t)$ satisfies the equation

$$\sigma_B\left(-i\frac{1}{\varepsilon}\frac{\partial v}{\partial t}\right) - \sigma_A\left(-i\frac{1}{\delta}\frac{\partial v}{\partial x}\right) + \widetilde{\gamma} v = 0. \tag{9}$$

The equations (8) and (9) are the matrix wave equations. Let us consider now the more general collective motions from the form

$$T(x,t) = e^{i(\varepsilon tA+\delta xB)}, \quad T^{*-1}(x,t) = e^{i(\overline{\varepsilon}tA^*+\overline{\delta}xB^*)}. \tag{10}$$

It is evident that the vector function (or so-called open field, following M.S. Livšic) $f(x,t) = T(x,t)h$ $(h \in H)$ satisfies the system (7) with identically zero input and an arbitrary initial state $f(0,0) = h$ $(h \in H)$.

Theorem 2. *(See [3].) The operator functions*

$$V(x,t) = \Phi T(x,t) = \Phi e^{i(\varepsilon tA+\delta xB)}, \quad \widetilde{V}(x,t) = \Phi T^{*-1}(x,t) = \Phi e^{i(\overline{\varepsilon}tA^*+\overline{\delta}xB^*)}$$

are solutions of the next systems of the partial differential equations (or matrix wave equations)

$$\left(\sigma_B\left(-i\frac{1}{\varepsilon}\frac{\partial}{\partial t}\right) - \sigma_A\left(-i\frac{1}{\delta}\frac{\partial}{\partial x}\right) + \widetilde{\gamma}\right)V = 0, \tag{11}$$

$$\left(\sigma_B\left(-i\frac{1}{\overline{\varepsilon}}\frac{\partial}{\partial t}\right) - \sigma_A\left(-i\frac{1}{\overline{\delta}}\frac{\partial}{\partial x}\right) + \gamma\right)\widetilde{V} = 0. \tag{12}$$

Let now the operator B be the triangular model of a coupling of a dissipative operator and an antidissipative operator with absolutely continuous real spectra (introduced in [1] and investigated in [5]).

$$\begin{aligned} Bf(w) = \alpha(w)f(w) - i\int_{a'}^{w} f(\xi)\Pi(\xi)S^*\Pi^*(w)d\xi + i\int_{w}^{b'} f(\xi)\Pi(\xi)S\Pi^*(w)d\xi + \\ +i\int_{a'}^{w} f(\xi)\Pi(\xi)L\Pi^*(w)d\xi, \end{aligned} \tag{13}$$

where $f = (f_1, f_2, \ldots, f_p) \in H = \mathbf{L}^2(\Delta;\mathbb{C}^p)$, $\Delta = [a', b']$, $L : \mathbb{C}^m \longrightarrow \mathbb{C}^m$, $\det L \neq 0$, $L^* = L$, $L = J_1 - J_2 + S + S^*$,

$$J_1 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}, \; S = \begin{pmatrix} 0 & 0 \\ \widehat{S} & 0 \end{pmatrix}, \tag{14}$$

r is the number of the positive eigenvalues and $m-r$ is the number of the negative eigenvalues of the matrix L, $\Pi(w)$ is a measurable $p \times m$ $(1 \leq p \leq m)$ matrix function on Δ, whose rows are linearly independent at each point of a set of positive measure, the matrix function $\widetilde{\Pi}(w) = \Pi^*(w)\Pi(w)$ satisfies the conditions

$$tr\,\widetilde{\Pi}(w) = 1, \quad \widetilde{\Pi}(w)J_1 = J_1\widetilde{\Pi}(w),$$

$||\widetilde{\Pi}(w_1) - \widetilde{\Pi}(w_2)|| \leq C|w_1 - w_2|^{\alpha_1}$ for all $w_1, w_2 \in \Delta$ for some constant $C > 0$, α_1 is an appropriate constant with $0 < \alpha_1 \leq 1$ (see [5]), (where $||\quad||$ is the norm in $\mathbb{C}^m$) and the function $\alpha : \Delta \longrightarrow \mathbb{R}$ satisfies the conditions:

(i) the function $\alpha(w)$ is continuous strictly increasing on Δ;

(ii) the inverse function $\sigma(u)$ of $\alpha(w)$ is absolutely continuous on $[a, b]$ ($a = \alpha(a')$, $b = \alpha(b')$);

(iii) $\sigma'(u)$ is continuous and satisfies the relation $|\sigma'(u_1) - \sigma'(u_2)| \leq C|u_1 - u_2|^{\alpha_2}$, $(0 < \alpha_2 \leq 1)$ for all u_1, $u_2 \in [a, b]$ and for some constant $C > 0$.

In [5] it has been obtained the existence and in [2] it has been obtained the explicit form of the wave operators $W_\pm(B^*, B)$ in the strong sense for the couple (B^*, B), where B is the model of a coupling (13) and wave operators $W_\pm(B^*, B)$ for the couple (B^*, B) have the form

$$W_\pm(B^*, B) = s - \lim_{x\to\pm\infty} e^{ixB^*}e^{-ixB} = \widetilde{S}^*_\mp\widetilde{S}_\mp \tag{15}$$

onto the space $\mathbf{L}^2(\Delta; \mathbb{C}^p)$, where $\widetilde{S}_\mp$ are obtained in [5] in explicit form in terms of multiplicative integrals and a finite dimensional analogue of the classical gamma function (see, for example, [5, 6]).

Let us consider the case when $\delta = 1$ in collective motions (10). Then

$$V(x, t) = \Phi e^{i(xB+\varepsilon tA)} = \Phi T(x, t)$$

satisfies the equation (11) with $\delta = 1$, which can be written in the form

$$D\left(-i\frac{1}{\varepsilon}\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x}\right)V(x, t) = 0,$$

where $D(\lambda, \mu) = \det(\lambda\sigma_B - \mu\sigma_A + \gamma) = \det(\lambda\sigma_B - \mu\sigma_A + \widetilde{\gamma})$ is the discriminant polynomial of the pair of operators A, B. Then from Theorem 1 it follows that the corresponding output of the open system (7) (with $\delta = 1$) satisfies the following matrix partial differential equation

$$\sigma_B\left(-i\frac{1}{\varepsilon}\frac{\partial v}{\partial t}\right) - \sigma_A\left(-i\frac{\partial v}{\partial x}\right) + \widetilde{\gamma}v = 0. \tag{16}$$

Let $\widehat{H}$ be the principal subspace of the pair A, B, i.e. $\widehat{H} = \overline{span}\{A^kB^j\Phi^*p, k, j = 0, 1, 2, \ldots, p \in \mathbb{C}^m\}$. Let $\widetilde{H}$ be the set of solutions

$$v_h(x, t) = \Phi e^{i(xB+\varepsilon tA)}h, \qquad h \in \widehat{H} \tag{17}$$

of the equation (16). Let the operator $U : \widehat{H} \longrightarrow \widetilde{H}$ be defined by the equality

$$Uh = \Phi e^{i(xB+\varepsilon tA)}h = v_h(x, t), \qquad h \in \widehat{H}. \tag{18}$$

Then from the existence of the limit

$$\lim_{x\to+\infty}(e^{ixB}h, e^{ixB}h) = (\widetilde{S}^*_+\widetilde{S}_+h, h), \qquad h \in \widehat{H} \tag{19}$$

for the coupling B it follows that

$$(h,h)=\lim_{x\to+\infty}(e^{ixB}h,e^{ixB}h)+\int_0^\infty(\sigma_B\Phi e^{ixB}h,\Phi e^{ixB}h)dx. \tag{20}$$

Now the equality (20) shows that the formula

$$\langle v_{h_1}(x,t),v_{h_2}(x,t)\rangle=\lim_{x\to+\infty}(e^{ixB}h_1,e^{ixB}h_2)+\int_0^\infty(\sigma_B v_{h_1}(x,0),v_{h_2}(x,0))dx \tag{21}$$

defines a scalar product in $\widetilde{H}$ and the operator U is an isometric one. Now as in [4] it can be proved the next theorem

Theorem 3. *Let $X=(A,B,H,\Phi,E,\sigma_A,\sigma_B,\gamma,\widetilde{\gamma})$ be a commutative regular colligation with an operator B defined by (13). Let $\widehat{H}$ be the principal subspace of the pair A, B. Then the colligation*

$$\widehat{X}=(A,B,\widehat{H},\Phi,E,\sigma_A,\sigma_B,\gamma,\widetilde{\gamma})$$

is unitary equivalent to the colligation

$$\widetilde{X}=(\widetilde{A},\widetilde{B},\widetilde{H},\widetilde{\Phi},E,\sigma_A,\sigma_B,\gamma,\widetilde{\gamma})$$

where $\widetilde{A}=-i\frac{1}{\varepsilon}\frac{\partial}{\partial t}$, $\widetilde{B}=-i\frac{\partial}{\partial x}$, $\widetilde{H}$ is a set of solutions (18) of the equation (16) such that

1) $\widetilde{H}$ is a Hilbert space with respect to the scalar product (21);

2) if $v(x,t)$ belongs to $\widetilde{H}$ then $\widetilde{A}v(x,t)$ and $\widetilde{B}v(x,t)$ belong to $\widetilde{H}$;

3) the operators $\widetilde{A}$ and $\widetilde{B}$ are bounded in $\widetilde{H}$;

4) the next equality holds

$$\lim_{\xi\to+\infty}\langle e^{i\xi\widetilde{B}}v_h(x,t),e^{i\xi\widetilde{B}}v_h(x,t)\rangle=\lim_{\xi\to+\infty}(e^{i\xi\widetilde{B}}h,e^{i\xi\widetilde{B}}h)\quad(v_h\in\widetilde{H}).$$

The proof of the theorem follows from the existence of the limit (19), obtained in explicit form in [5].

Following the introduced definitions by M.S. Livšic in [7] the functions $v_h(x,t)$ and $v_h(x,0)$ are said to be the output realization and the mode of an element $h\in\widehat{H}$ correspondingly.

At first let us consider (7) in the case when $\varepsilon=1$ and $\widetilde{U}$ is an unitary operator such that $\widetilde{U}B\widetilde{U}^*=-B^*$, $\widetilde{U}A\widetilde{U}^*=-A^*$, $A=4B^3$, B is the considered coupling. Then

$$T(x,t)=e^{i(xB+4tB^3)},\quad T^{*-1}(x,t)=\widetilde{U}e^{-i(xB+4tB^3)}\widetilde{U}^*=\widetilde{U}T(-x,-t)\widetilde{U}^*$$

and the solitonic combination

$$S(x,t)=\Gamma^{-1}(x,t)\Gamma_x(x,t),$$

where $\Gamma_x(x,t)=T(x,t)N+T(-x,-t)\widetilde{M}$ takes the form

$$S(x,t)=\Gamma^{-1}\Gamma_x=iB-i\Gamma^{-1}N_1T(-x,-t)(BM_1+M_1B)$$

where the solitonic combination satisfies the operator KdV equation. Using results in [3] we have that $N_1 = N\widetilde{U}$,

$$M_2 = U^* M = \int_0^\infty e^{ixB} \rho_1 \frac{B - B^*}{i} e^{ixB} dx \tag{22}$$

where we have denoted $\rho_1 = \widetilde{U}^* \rho$ and the operator M is defined by the equality

$$M = \int_0^\infty e^{-ixB^*} \rho \frac{B - B^*}{i} e^{ixB} dx$$

which satisfies the equality

$$B^* M - MB = \rho(B^* - B).$$

(The last equality follows, using the asymptotic behaviour of the nondissipaive curves $e^{ixB} f$ as $x \pm \infty$, the form of $\Phi e^{ixB} f$ (see [5]) and obtained in [3].) Now using that $P_B \frac{\partial}{\partial x} S(x,t) P_B$ is the scalar solution of the nonlinear equation

$$v_t - 6 v v_x + v_{xxx} = 0.$$

Consequently we have to find the expression

$$h = -i\Gamma^{-1} N_1 T(-x, -t)(BM_1 + M_1 B) g, \quad g \in G_B.$$

Let now choose $N_1 = I$. Then h takes the form

$$h = (I + T(-2x, -2t) M_1)^{-1} T(-2x, -2t) \rho_1 \Phi^* \sigma_B \Phi g. \tag{23}$$

Hence

$$h = -T(-2x, -2t) M_1 h + T(-2x, -2t) \rho_1 \Phi^* \sigma_B \Phi g. \tag{24}$$

But from (22) and (17) we have

$$M_1 h = \int_0^\infty e^{i\eta B} \rho_1 \Phi^* \sigma_B v_h(\eta, 0) d\eta. \tag{25}$$

Now (24) and (25) imply that

$$h = -T(-2x, -2t) \int_0^\infty e^{i\eta B} \rho_1 \Phi^* \sigma_B v_h(\eta, 0) d\eta + T(-2x, -2t) \rho_1 \Phi^* \sigma_B \Phi g. \tag{26}$$

Consequently, from (26) and (17) it follows that

$$\widehat{h}(\xi) = v_h(\xi, 0) = \Phi e^{i\xi B} h =$$
$$= -\Phi e^{i\xi B} T(-2x, -2t) \int_0^\infty e^{i\eta B} \rho_1 \Phi^* \sigma_B v_h(\eta, 0) d\eta + \Phi e^{i\xi B} T(-2x, -2t) \rho_1 \Phi^* \sigma_B v_g(0, 0)$$

and hence

$$\widehat{h}(\xi) = v_h(\xi, 0) = \Phi e^{i\xi B} h =$$
$$= -\Phi \int_0^\infty T(\xi + \eta - 2x, -2t) \rho_1 \Phi^* \sigma_B \widehat{h}(\eta) d\eta + \Phi T(\xi - 2x, -2t) \rho_1 \Phi^* \sigma_B v_g(0, 0)$$

or the equivalent representation

$$\begin{aligned} \widehat{h}(\xi) = v_h(\xi, 0) = \Phi e^{i\xi B} h = \\ = -\int_0^\infty \Psi(\xi + \eta - 2x, -2t)\widehat{h}(\eta) d\eta + \Psi(\xi - 2x, -2t) v_g(0, 0), \end{aligned} \tag{27}$$

where we have denoted

$$\Psi(x, t) = \Phi T(x, t)\rho_1 \Phi^* \sigma_B.$$

The equation (27) is a generalized form of the Gelfand-Levitan-Marchenko equation, presented in terms of the output realization of the two-operator colligation. Consequently, we have proved the next theorem:

Theorem 4. *Let the operator B be the coupling (13) of dissipative and antidissipative operators with absolutely continuous real spectra. Let the operators $A = 4B^3$ be embedden in a regular colligation from the form (1) and B satisfies the condition $B^* = -\widetilde{U}B\widetilde{U}^*$, where $\widetilde{U}$ is unutary operator in H. Then the generalized Gelfand-Levitan-Marchenko equation for the KdV equation is*

$$\begin{aligned} \widehat{h}(\xi) = v_h(\xi, 0) = \Phi e^{i\xi B} h = \\ = -\int_0^\infty \Psi(\xi + \eta - 2x, -2t)\widehat{h}(\eta) d\eta + \Psi(\xi - 2x, -2t) v_g(0, 0), \end{aligned}$$

where

$$\Psi(x, t) = \Phi T(x, t)\rho_1 \Phi^* \sigma_B$$

and $v_h(x, t)$ is the output realization (17).

Finally it is worth to mention that analogously it can be considered different cases, corresponding to the Schrödinger equation, the Heisenberg equation, the Sine-Gordon equation, which relate to the different values of the constant ε, and the Davey-Stewartson equation (with the help of the connection between three commuting nonselfadjoint operators and the soliton theory).

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