

ON THE CAUCHY PROBLEM FOR THE STRONG DISPERSIVE NONLINEAR WAVE EQUATION

MEHMED I. KODZHA

ABSTRACT: *In this paper we consider the Cauchy problem for the strong dispersive nonlinear wave equation. We proof that the operator of the linearization arround the self-similar solutions generate C_0-semigroup in Sobolev space $\mathbb{H}^s$. Global existence is obtained via Banach fixed point theorem.*

KEYWORDS: *Nonlinear dispersive equation, Cauchy problem*

DOI: https://doi.org/10.46687/GMKW7508

1 Introduction

In this paper we consider the following strong dispersive nonlinear wave equation

$$u_t - \alpha^2 u_{txx} + 2ku_x + 3uu_x + \gamma(u - \alpha^2 u_{xx})_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}). \tag{1.1}$$

Equation (1.1) is a version of the following well-known generalization of the Dullin-Gottwald-Holm equation [1]

$$u_t - \alpha^2 u_{txx} + 2\omega u_x + 3uu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}). \tag{1.2}$$

Equation (1.2) is derived in [1] as a model for shallow water waves. The Cauchy problem for the equation Dullin-Gottwald-Holm in both periodic and non periodic case was studied in [5, 7, 8]. For (1.2) the problem of the asymptotic stability of self-similar solution was considered in [4]. The authors construct explicit self-similar solutions and consider the dynamical behavior of the solutions near to the self-similar solutions. Moreover the asymptotic stability also was considered.

For Eq. (1.1) self-similar solutions are in the form

$$\overline{u}(t,x) = -\frac{1}{3}\left(\frac{x}{T-t} + 2k\right)$$

and dynamical behavior arround of these solutions was considered in [3]. Consider the perturbation of the form

$$u(t,x) = v(t,x) + \overline{u}(t,x).$$

For v, we get the following equation

$$\begin{aligned} &v_\tau - \alpha^2 e^{2\tau} v_{\tau\rho\rho} - \tfrac{4\alpha^2}{3} e^{2\tau} v_{\rho\rho} + e^{2\tau}\left(\gamma + \tfrac{2\alpha^2}{3}(k-\rho)\right) v_{\rho\rho\rho} \\ &-\alpha^2\gamma e^{4\tau} v_{\rho\rho\rho\rho\rho} - v + 3vv_\rho = \alpha^2 e^{2\tau}(2v_\rho v_{\rho\rho} + vv_{\rho\rho\rho}), \end{aligned} \tag{1.3}$$

where $\tau = -log(T-t)$, $\rho = \frac{x}{T-t}$ and $T > 0$.

Equation (1.1) is can be written as a dissipative non-local equation

$$\begin{aligned} &w_\tau + \tfrac{1}{3}w - e^{-\tau}\left(\tfrac{2k+e^\tau \rho_0}{3}\right) w_{\rho_0} + \gamma e^{-\tau} w_{\rho_0\rho_0\rho_0} - \tfrac{1}{3}(p\star w) \\ &+e^{-\tau}\left(\tfrac{2(k-e^\tau\rho_0)}{3}\right)(p\star w)_{\rho_0} + 3(p\star w)(p\star w)_{\rho_0} \\ &= 2(p\star w)_{\rho_0}(p\star w - w) + (p\star w)((p\star w)_{\rho_0} - w_{\rho_0}), \end{aligned} \tag{1.4}$$

where $w(\tau,\rho_0)=\bar{v}(\tau,\rho_0)-\alpha^2\bar{v}_{\rho_0\rho_0}(\tau,\rho_0)$, $\bar{v}(\tau,\rho_0)=e^{-\tau}v(\tau,\rho)$ *and* $\rho_0:=e^{-\tau}\rho$ with the initial data

$$\begin{aligned} w(0,\rho_0):=w_0(\rho_0) &= v_0(\rho_0)-\alpha^2 v_{\rho_0\rho_0}(0,\rho_0) \\ &= u_0(x)-\alpha^2 u_0^{||}(x)+\tfrac{1}{3}\left(\tfrac{x}{T}+2k\right) \end{aligned} \tag{1.5}$$

and the boundary conditions

$$\lim_{|\rho_0|\to+\infty} w(\tau,\rho_0)=0, \quad \lim_{|\rho_0|\to+\infty} w_{\rho_0}(\tau,\rho_0)=0. \tag{1.6}$$

Our aim in this paper to study the global well-posedness for Eq. (1.4) with the initial data (1.5) and boundary conditions (1.6).

2 Main result

In this section we study the global well-posedness for Eq. (1.4) with the initial data (1.5) and boundary conditions (1.6). We denote by L the following linear operator

$$\begin{aligned} L[w]:= & -\tfrac{1}{3}w+e^{-\tau}\tfrac{1}{3}(2k+e^{\tau}\rho_0)w_{\rho_0}-\gamma e^{-\tau}w_{\rho_0\rho_0\rho_0} \\ & +\tfrac{1}{3}(p\star w)-e^{-\tau}\tfrac{2}{3}(k-e^{\tau}\rho_0)(p\star w)_{\rho_0}, \end{aligned} \tag{2.1}$$

then Eq. (1.4) can be rewritten as

$$w_\tau = L[w]+f(w), \tag{2.2}$$

where the nonlinear term

$$f(w):=-3(p\star w)(p\star w)_{\rho_0}+2(p\star w)_{\rho_0}(p\star w-w)+(p\star w)[(p\star w)_{\rho_0}-w_{\rho_0}].$$

Lemma 2.1. *For $s>4$, we have*

- $L[w]\in\mathbb{H}^s, \quad \forall w\in\mathscr{D}(L)$.
- *L is a closed and densely defined linear operator in $\mathbb{H}^s$.*

Proof. It is a direct verification based on the definition of L in (2.1). □

Lemma 2.2. *For $s>4$, L is a dissipative operator in $\mathbb{H}^s$, i.e., $(L[w],w)_s\le 0$.*

Proof. A direct calculations leads to

$$\begin{aligned} & \textstyle\int_{\mathbb{R}}(\Lambda^s L[w])\Lambda^s w d\rho_0 \\ = & -\tfrac{1}{3}\|w\|_{\mathbb{H}^s}^2+\tfrac{1}{3}\textstyle\int_{\mathbb{R}}\Lambda^s w\Lambda^s(p\star w)d\rho_0 \\ & +\textstyle\int_{\mathbb{R}}\Lambda^s w\Lambda^s\left[e^{-\tau}\frac{2k+e^{\tau}\rho_0}{3}w_{\rho_0}\right]d\rho_0-\gamma e^{-\tau}\int_{\mathbb{R}}\Lambda^s w\Lambda^s[w_{\rho_0\rho_0\rho_0}]d\rho_0 \\ & -\textstyle\int_{\mathbb{R}}\Lambda^s w\Lambda^s\left\{\left[e^{-\tau}\frac{2(k-e^{\tau}\rho_0)}{3}\right](p\star w)_{\rho_0}\right\}d\rho_0. \end{aligned}$$

As in Lemma 3.2. [3], we get

$$\begin{aligned}
&\int_{\mathbb{R}}(\Lambda^s L[w])\Lambda^s w d\rho_0 \\
=\ & -\tfrac{1}{3}\|w\|^2_{\mathbb{H}^s} + \tfrac{1}{3}\|w\|^2_{\mathbb{H}^{s-1}} - \tfrac{1}{6}\|w\|^2_{\mathbb{H}^s} - \|w\|^2_{\mathbb{H}^{s-1}} \\
=\ & -\tfrac{1}{2}\|w\|^2_{\mathbb{H}^s} - \tfrac{2}{3}\|w\|^2_{\mathbb{H}^{s-1}} \leq 0.
\end{aligned}$$

This completes the proof. □

Lemma 2.3. *Let $s > 4$. Then the operator L is invertible in $\mathbb{H}^s$. Furthermore, it generates a C_0-semigroup $(\boldsymbol{S}(\tau))_{\tau\geq 0}$ in $\mathbb{H}^s$.*

Proof. First, we will show that L^{-1} exist. We need to prove that L is injective and surjective. Let $w \in \mathscr{D}(L)$, such that $L[w] = 0$. Then

$$\begin{aligned}
& -\tfrac{1}{3}\Lambda^s w + \tfrac{1}{3}\Lambda^s(p\star w) + \Lambda^s\left[e^{-\tau\frac{2k+e^\tau\rho_0}{3}} w_{\rho_0}\right] \\
& -\gamma\Lambda^s[e^{-\tau}w_{\rho_0\rho_0\rho_0}] - \Lambda^s\left\{\left[e^{-\tau\frac{2(k-e^\tau\rho_0)}{3}}\right](p\star w)_{\rho_0}\right\} = 0.
\end{aligned} \tag{2.3}$$

Multiplying (2.3) by $\Lambda^s w$ and integrating over $\mathbb{R}$, we get

$$\int_{\mathbb{R}} \Lambda^s L[w]\Lambda^s w d\rho_0 = -\frac{1}{2}\|w\|^2_{\mathbb{H}^s} - \frac{2}{3}\|w\|^2_{\mathbb{H}^{s-1}} = 0.$$

This combining with the boundary conditions (1.6) gives that $w = 0$. So the operator L is injective. On the other hand, for all $g \in \mathbb{H}^1$ consider the equation

$$L[w] = g. \tag{2.4}$$

Applying Λ^s to (2.4) and multiplying the result by $\Lambda^s w$, and then integrating over $\mathbb{R}$, we get

$$\|w\|^2_{\mathbb{H}^s} \leq \|w\|^2_{\mathbb{H}^s} + \frac{4}{3}\|w\|^2_{\mathbb{H}^{s-1}} = -2\int_{\mathbb{R}} \Lambda^s g \Lambda^s w d\rho_0.$$

It follows from the Young's inequality that

$$\|w\|_{\mathbb{H}^s} \leq C\|g\|_{\mathbb{H}^s}.$$

Note that $s > 4$, then by the standard theory of elliptic-type partial differential equations, there exists a unique weak solution $w \in \mathbb{H}^1$, moreover, we have $w \in \mathbb{H}^{s+1}$ if $g \in \mathbb{H}^s$. Thus, the operator L is surjective.

Secondly, by the Lumer-Phillips theorem [6], we obtain that the operator L generates a C_0-semigroup $(\mathbf{S}(\tau))_{\tau\geq 0}$ in $\mathbb{H}^s$. This completes the proof. □

As a consequence, the results of Lemma 2.1-2.3 imply that the following existence theorem holds.

Proposition 2.1. *Let $s > 2$. Then the Cauchy problem*

$$\begin{cases} \frac{d}{d\tau} w = Lw, \\ w(0) = w_0 \end{cases}$$

with zero boundary condition at far field exists a unique solution $w(\tau) = \boldsymbol{S}(\tau) w_0$, where w_0 is the initial data defined in (1.5).

Note that Proposition 2.1 combining with the Duhamel's principle yield that the solutions of Eq. (2.2) satisfying the following integral equation:

$$w(\tau) = \mathbf{S}(\tau + lnT) w_0 + \int_{-lnT}^{\tau} \mathbf{S}(\tau - s) f(w(s)) ds \ \text{ for } \ \tau \geq -lnT.$$

In order to remove the dependence of the integral equation on the blow up time T, we introduce a new variable $\widehat{w}$ defined in $[0, +\infty)$ by

$$\widehat{w}(\tau) = w(\tau - lnT). \tag{2.5}$$

Thus, the above integral equation is equivalent to

$$\widehat{w}(\tau) = \mathbf{S}(\tau) w_0 + \int_0^{\tau} \mathbf{S}(\tau - s) f(\widehat{w}(s)) ds \ \text{ for } \ \tau \geq 0.$$

For convenience, we take $\widehat{}$ away and rewrite it as

$$w(\tau) = \mathbf{S}(\tau) w_0 + \int_0^{\tau} \mathbf{S}(\tau - s) f(w(s)) ds \ \text{ for } \ \tau \geq 0.$$

To show this integral equation exists a solution, we define the closed ball in $\mathbb{H}^s$ ($s > 2$) as follows:

$$B_\delta = \{w \in \mathbb{H}^s : \|w\|_{\mathbb{H}^s} < \delta \ll 1\}.$$

Define the map $\mathscr{T}$ as

$$\mathscr{T} w(\tau) = \mathbf{S}(\tau) w_0 + \int_0^{\tau} \mathbf{S}(\tau - s) f(w(s)) ds.$$

We need to prove that $\mathscr{T}$ has a fixed point in the space B_δ for some $\delta < 1$ by using the Banach fixed point theorem. To achieve this, we introduce the following inequality.

Lemma 2.4. *([2]) Let $s > 2$. Then $\mathbb{H}^s \cap \mathbb{L}^\infty$ is an algebra, and*

$$\|uv\|_{\mathbb{H}^s} \leq C(\|u\|_{\mathbb{L}^\infty} \|v\|_{\mathbb{H}^s} + \|u\|_{\mathbb{H}^s} \|v\|_{\mathbb{L}^\infty}),$$

where C is a positive constant depending upon s.

Lemma 2.5. *Let $s > 2$ be an integer. Assume that $\|w_0\|_{\mathbb{H}^{s+1}} < \delta$ for some sufficiently small $\delta > 0$. Then $\mathscr{T}$ is a self-mapping on B_δ. Moreover, $\mathscr{T}$ is a contraction mapping.*

Proof. By Lemma 2.4, we have

$$\begin{aligned}
\|f(w)\|_{\mathbb{H}^s} &\leq 3\|(p\star w)(p\star w)_{\rho_0}\|_{\mathbb{H}^s} \\
&\quad +2\|(p\star w)_{\rho_0}(p\star w-w)\|_{\mathbb{H}^s}+\|(p\star w)((p\star w)_{\rho_0}-w_{\rho_0})\|_{\mathbb{H}^s} \\
&\leq \|p\star w\|_{\mathbb{H}^s}\|(p\star w)_{\rho_0}\|_{\mathbb{L}^\infty}+\|(p\star w)_{\rho_0}\|_{\mathbb{L}^\infty}\|(p\star w-w)\|_{\mathbb{H}^s} \\
&\quad +\|p\star w\|_{\mathbb{H}^s}\|(p\star w)_{\rho_0}-w_{\rho_0}\|_{\mathbb{L}^\infty}.
\end{aligned}$$

Note that $\mathbb{H}^s \subset \mathbb{L}^\infty$ and $w=\Lambda^2(p(\rho_0)\star\bar{v})$, then using Lemma 3.2 [3], we have

$$\|f(w)\|_{\mathbb{H}^s} \leq C\|w\|^2_{\mathbb{H}^s} < C\delta^2 < \delta$$

for sufficiently small δ. Thus, $\mathscr{T}$ is a self-mapping on B_δ.

To show that $\mathscr{T}$ is a contraction mapping, we choose $w,\overline{w}\in B_\delta$, by Lemma 2.4 and a direct calculation shows that

$$\begin{aligned}
\|f(w)-f(\overline{w})\|_{\mathbb{H}^s} &\leq 3\|(p\star w)(p\star w)_{\rho_0}-(p\star\overline{w})(p\star\overline{w})_{\rho_0}\|_{\mathbb{H}^s} \\
&\quad +2\|(p\star w)_{\rho_0}(p\star w-w)-(p\star\overline{w})_{\rho_0}(p\star\overline{w}-\overline{w})\|_{\mathbb{H}^s} \\
&\quad +\|(p\star w)((p\star w)_{\rho_0}-w_{\rho_0})-(p\star\overline{w})((p\star\overline{w})_{\rho_0}-\overline{w}_{\rho_0})\|_{\mathbb{H}^s} \\
&\leq 3\|(p\star(w-\overline{w}))(p\star w)_{\rho_0}\|_{\mathbb{H}^s}+3\|(p\star\overline{w})(p\star(w-\overline{w}))_{\rho_0}\|_{\mathbb{H}^s} \\
&\quad +2\|(p\star(w-\overline{w}))_{\rho_0}(p\star w-w)\|_{\mathbb{H}^s} \\
&\quad +2\|(p\star\overline{w})_{\rho_0}(p\star(w-\overline{w})-(w-\overline{w}))\|_{\mathbb{H}^s} \\
&\quad +\|(p\star(w-\overline{w}))((p\star w)_{\rho_0}-w_{\rho_0})\|_{\mathbb{H}^s} \\
&\quad +\|(p\star\overline{w})((p\star(w-\overline{w}))_{\rho_0}-(w-\overline{w}_{\rho_0}))\|_{\mathbb{H}^s} \\
&\leq C\delta\|w-\overline{w}\|_{\mathbb{H}^s}.
\end{aligned}$$

Thus,

$$\|\mathscr{T}w(\tau)-\mathscr{T}\overline{w}(\tau)\|_{\mathbb{H}^s} \leq C\delta\|w-\overline{w}\|_{\mathbb{H}^s}.$$

Since $\delta>0$ is sufficiently small, $\mathscr{T}$ is a contraction mapping. □

We now return to the existence of solution for nonlinear Eq. (1.3).

Theorem 2.1. *Let integer $s>4$. The nonlinear Eq. (1.3) with the initial data (1.5) and boundary conditions (1.6) admits a global solution $v(\tau,\rho)\in\mathbb{H}^s$. Moreover, if the initial data $\|v_0\|_{\mathbb{H}^{s+1}}<\delta$ for some sufficiently small $\delta>0$, then*

$$\|v\|_{\mathbb{H}^s} \leq C\delta e^{-2\tau}.$$

Proof. By Lemma 2.5 and Banach fixed point theorem the map $\mathscr{T}$ has a fixed point in B_δ. The fixed point is the solution of equation (2.2) and equation (1.3) has a global solution

$$v(\tau,\rho) = e^{\tau}\bar{v}(\tau,\rho_0) = e^{\tau}(p \star w(\tau,\rho_0)), \tag{2.6}$$

where $w(\tau,\rho_0)$ is a global solution of Eq. (2.2) given in Lemma 2.5, and $\rho_0 = e^{-\tau}\rho$.

Furthermore, it follows from (2.6), that $v_{\rho\rho} = e^{-\tau}w$. From Lemma 3.2 [3], we get

$$\|v\|_{\mathbb{H}^s} \le e^{-\tau}\|w\|_{\mathbb{H}^{s-2}} \le Ce^{-2\tau}\|w_0\|_{\mathbb{H}^{s-2}} < C\delta e^{-2\tau}.$$

□

Acknowledgments

This paper is partially supported by Scientific Research Grant RD-08-28/12.01.2022 of Shumen University.

REFERENCES:

[1] Dullin H, Gottwald G, Holm D. An integrable shallow water equation with linear and nonlinear dispersion. Phy Rev Lett 2001; 87:194501. 4

[2] Kato T. Quasi-linear equations of evolution, with applications to partial differential equations, in: Spectral theory and differential equations (proc. sympos.), Dundee, 1974. In: Lecture notes in Math., vol. 48. Berlin: Springer; 1975. p. 25-70. Dedicated to Konrad Jorgens

[3] Kodzha M. Dynamical behavior of the strong dispersive nonlinear wave equation. Annual of Konstantin Preslavsky University of Shumen, Vol. XXII C, 2021, pp. 3-11.

[4] Li H, Yan W. Asyptotic stability and instability of explicit self-similar waves for a class of nonlinear shallow water equations. Commun Nonlinear Sci Numer Simulat 79 (2019); 104928

[5] Liu Y. Global existence and blowup solutions for a nonlinear shallow water equation. Math Ann 2006; 335:717-35.

[6] Pazy A. Semigroups of linear operators and applications to partial differential equations. New York: Springer-Verlag; 1983.

[7] Tian LX, Gui GL, Liu Y. On the well-posedness problem and the scattering problem for the Dullin-Gottwald-Holm equation. Commun Math Phys 2005; 257:667-70.

[8] Zhou Y. Blow-up of solutions to the DGH equation. J Functional Anal 2007; 250:227-48.