A NEW ALGORITHM FOR DYNAMIC MODE DECOMPOSITION*

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ABSTRACT: *Dynamic mode decomposition (DMD) is a data-driven mathematical technique to extract spectral information from complex data coming from numerical or experimental studies of various systems. It is an equation-free method in the sense that it does not require knowledge of the underlying governing equations. In this article we explore and demonstrate a new algorithm for calculating the DMD decomposition.*

KEYWORDS: *Dynamic mode decomposition, Koopman operator, Singular value decomposition, Equation-free, Frobenius companion matrix*

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1 Introduction

Introduced for the first time by Schmid [1] in the fluid mechanics community, *dynamic mode decomposition* (DMD) is a method for analyzing data from numerical simulations and laboratory experiments. The method constitutes a mathematical technique for identifying spatiotemporal coherent structures from high-dimensional data. It combines the favorable features from two of the most powerful data analytic tools: Proper orthogonal decomposition (POD) in space and Fourier transforms in time. It is relatively easy to implement and makes no additional assumptions about the underlying dynamics. DMD method can be considered to be a numerical approximation to Koopman spectral analysis, and in this sense it is applicable to nonlinear dynamical systems (see [2, 3]). It is an equation-free in the sense that it does not require knowledge of the underlying governing equations and it is entirely data-driven.

DMD method has application in many different fields such as video processing [12], epidemiology [13], neuroscience [15], financial trading [16, 17, 18], robotics [14], cavity flows [4, 6] and various jets [2, 5]. For a review of the DMD literature, we refer the reader to [7, 8, 9, 19].

The remainder of this work is organized as follows: in the rest of Section 1 we describe the DMD algorithm, in Section 2, we explore and discuss a new approach for DMD computation and in Section 3 we present example demonstrating the new algorithm. The conclusion is in Section 4.

2 Dynamic Mode Decomposition

We follow the definition of DMD method introduced in [8]. Suppose we have two data sets

$$X = [x_0, \dots, x_m] \quad \text{and} \quad Y = [y_0, \dots, y_m], \tag{1}$$

such that $y_k = f(x_k)$, where f is a map associated with the evolution of a dynamical system

$$\dot{y}(x,t) = f(y(x,t)). \tag{2}$$

Assume that there exist a linear map A such that

$$Y \approx AX. \tag{3}$$

The DMD method computes the eigendecomposition of the best fit linear operator A. The eigenvectors of A are called *DMD modes* and each mode corresponds to a particular eigenvalue of A, also called *DMD eigenvalue*.

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We can approximate the operator A by using Eq.(3) and singular value decomposition of $X = U\Sigma V^*$ in a least-squares sense

$$A \approx YX^{\dagger} = YV\Sigma^{-1}U^*, \tag{4}$$

where $X^{\dagger}$ is the pseudoinverse of X. This solution is the minimizer of

$$\|Y - AX\|_F$$

in the case of over-determined $AX = Y$ and A is the minimum norm solution, of $AX = Y$ in the case that the equation is under-determined, where $\|.\|_F$ denotes the standard Frobenius norm.

It should be noted that if n is large, i.e. $n \gg 1$, calculating the eigendecomposition of the $n \times n$ matrix A can be prohibitively expensive. Therefore, the goal is to calculate eigenvectors and eigenvalues without explicit representation or direct manipulations of A. The *DMD modes* and *eigenvalues* are intended to approximate the eigenvectors and eigenvalues of A.

We can derive from (4) a low-dimensional representation (if reduced SVD of X is performed)

$$\tilde{A} = U^*AU = U^*YV\Sigma^{-1} \tag{5}$$

by using similarity transformation with matrix U. In practice, we calculate the eigendecomposition of reduced order matrix $\tilde{A}$.

Exact DMD algorithms

The following algorithm provides a robust method for computing DMD modes and eigenvalues.

Algorithm 1: Exact DMD

Input: Data matrices X and Y, and rank r.
Output: DMD modes Φ and eigenvalues Λ

1: **Procedure** DMD(X,Y,r).
2: $[U, \Sigma, V] = SVD(X, r)$ *(Reduced r-rank SVD of X)*
3: $\tilde{A} = U^*YV\Sigma^{-1}$ *(Low-rank approximation of A)*
4: $[W, \Lambda] = EIG(\tilde{A})$ *(Eigen-decomposition of $\tilde{A}$)*
5: $\Phi = YV\Sigma^{-1}W$ *(DMD modes of A)*
6: **End Procedure**

The standard definition of DMD method assumes a sequential set of data vectors $\{z_k\}_{k=0}^m$, rather than as a set of pairs $\{(x_k, yk)\}_{k=1}^m$. Note that the original formulation is a special case of (1), with $x_k = z_{k-1}$ and $y_k = z_k$.

The algorithm of the DMD method introduced in [1] differs slightly from the one described above. The only difference is that the DMD modes (at step 5) are computed by the formula

$$\Phi = UW. \tag{6}$$

The DMD modes calculated by *Algorithm 1* are called *exact DMD modes*, because Tu et al. in [8] prove that these are exact eigenvectors of matrix A. The modes computed by (6) are referred to as *projected DMD modes*.

Finally, knowing the DMD modes ϕ_i and eigenvalues λ_i, a continuous solution of associated with (2), locally linear system

$$\dot{y}(t) = Ay(t)\,, \quad \text{with initial condition} \;\; \mathbf{x}_0 = \mathbf{x}(0) \tag{7}$$

can be constructed as a function of time via equation

$$\mathbf{y}(t) = \Phi \exp(\Omega t)\mathbf{b}\,, \tag{8}$$

where $\Phi = [\phi_1, \ldots, \phi_r]$, $\Omega = diag\{\omega_1, \ldots, \omega_r\}$, $\mathbf{b} = [b_1, \ldots, b_r]^T$. Vector $\mathbf{b}$ consists the coefficients of the initial condition $\mathbf{x}_0$ in the eigenvector basis, so that $\mathbf{x}_0 = \Phi\mathbf{b}$. Continuous time eigenvalues (Fourier modes) ω_j can be converted from the DMD discrete-time eigenvalues λ_j via

$$\omega_j = \frac{\ln(\lambda_j)}{\triangle t}\,, \; j = 1, \ldots, r\,. \tag{9}$$

From the expression (8) a prediction of the future state of the system is achieved for any time t. A direct result of the formulation of the expansion of the solution as in (8) is that one now has access to characteristic spatiotemporal features of the system. The rate of growth/decay and frequency of oscillations of each DMD mode is given by the eigenvalue ω_j and the time dependent term $exp(\omega_j t)$ gives us the dynamics associated to each mode ϕ_j scaled with a constant b_j.

3 Alternatives of *DMD algorithm*

We will first describe a new algorithm for computing the DMD modes and eigenvalues of the linear operator A, introduced in [20].

An alternative of *exact DMD algorithm*

The DMD algorithm presented in previous section use the advantage of low dimensionality in the data to make a low-rank approximation of the operator A that best approximates the nonlinear dynamics of the data set. The main idea of new algorithm is to extract the modal structures from matrix

$$\hat{A} = \Sigma^{-1} U^* Y V\,, \tag{10}$$

instead from the matrix $\tilde{A}$ defined by (5). The two matrices $\tilde{A}$ and $\hat{A}$ are similar, with transformation matrix Σ, therefore have the same eigenvalues. Let

$$\hat{A}\hat{W} = \hat{W}\Lambda \tag{11}$$

be eigendecomposition of matrix $\hat{A}$. Then, using relations (4) and (11) we can easily obtain the following expression

$$AU\Sigma\hat{W} = U\Sigma\hat{W}\Lambda\,,$$

which yields the formula

$$\hat{\Phi} = U\Sigma\hat{W}\,, \tag{12}$$

for the DMD modes. Expression (12) corresponds to projected DMD modes defined by (6). It can be shown that matrix

$$\hat{\Phi} = YV\hat{W}, \tag{13}$$

corresponds to exact DMD modes $\Phi = YVW\Sigma^{-1}$ defined at Step 5 in Algorithm 1, (see Theorem 1 below).

Now, we are ready to formulate an alternative to exact DMD method described in Algorithm 1, see Algorithm 2 below.

Algorithm 2: Alternative exact DMD

Input: Data matrices X and Y, and rank r.
Output: DMD modes $\hat{\Phi}$ and eigenvalues Λ

1: **Procedure** DMD(X,Y,r).	
2: $[U,\Sigma,V] = SVD(X,r)$	*(Reduced r-rank SVD of X)*
3: $\hat{A} = \Sigma^{-1}U^*YV$	*(Low-rank approximation of A)*
4: $[\hat{W},\Lambda] = EIG(\hat{A})$	*(Eigen-decomposition of $\hat{A}$)*
5: $\hat{\Phi} = YV\hat{W}$	*(DMD modes of A)*
6: **End Procedure**	

Although from a computational point of view the matrices $\hat{A}$ and $\tilde{A}$ are similar, because have the same multipliers, calculation of DMD modes $\hat{\Phi}$ in Algorithm 2 is more cost efficient than the calculation of matrix Φ in Algorithm 1. The following theorem holds.

Theorem 1. *[20] Let (λ, w), for $\lambda \neq 0$, be an eigenpair of $\hat{A}$ defined by (10), then the corresponding eigenpair of A is $(\lambda, \hat{\varphi})$, where*

$$\hat{\varphi} = YVw.$$

An alternative of *projected DMD algorithm*

We can formulate second alternative to DMD method by using Eq.(12), see Algorithm 3 below.

In the following theorem we will prove that eigenvectors defined by (12) are the projected DMD modes.

Theorem 2. *Let (λ, w), for $\lambda \neq 0$, be an eigenpair of $\hat{A}$ defined by (10), and let P_X denotes the orthogonal projection matrix onto the column space of X. Then the vector*

$$\varphi = U\Sigma w$$

is an eigenvector of P_XA with eigenvalue λ. Furthermore, if $\hat{\varphi} = YVw$ is given by (13), then $P_X\hat{\varphi} = \lambda\varphi$.

Algorithm 3: Alternative projected DMD

Input: Data matrices X and Y, and rank r.
Output: DMD modes $\hat{\Phi}$ and eigenvalues Λ

1: **Procedure** DMD(X,Y,r).
2: $[U,\Sigma,V] = SVD(X,r)$ *(Reduced r-rank SVD of X)*
3: $\hat{A} = \Sigma^{-1}U^*YV$ *(Low-rank approximation of A)*
4: $[\hat{W},\Lambda] = EIG(\hat{A})$ *(Eigen-decomposition of $\hat{A}$)*
5: $\hat{\Phi} = U\Sigma\hat{W}$ *(DMD modes of A)*
6: **End Procedure**

Proof. From the SVD $X = U\Sigma V$, the orthogonal projection onto the column space of X is given by $P_X = UU^*$. From (5), (10) and the presentation

$$\Sigma^{-1}U^*AU\Sigma = \hat{A}$$

we get

$$P_XA\varphi = UU^*AU\Sigma w = U\Sigma(\Sigma^{-1}U^*AU\Sigma)w = U\Sigma\hat{A}w = \lambda U\Sigma w = \lambda\varphi\,.$$

From last expression it follows that φ is an eigenvector of P_XA with eigenvalue λ. Let now express

$$P_X\hat{\varphi} = UU^*YVw = U\Sigma(\Sigma^{-1}U^*YV)w = U\Sigma\hat{A}w = \lambda U\Sigma w = \lambda\varphi\,.$$

The proof is completed.

4 Numerical illustrative example

To illustrate Algorithm 3 introduced in Section 3, we consider a simple example of two mixed spatiotemporal signals (presented in [1, 8]).

Let us have two signals of interest

$$f_1(x,t) = \text{sech}(x+6-t)\exp(i2.3t)$$

and

$$f_2(x,t) = 2\text{sech}(x)\tanh(x)\exp(i2.8t),$$

and the mixed signal is

$$f(x,t) = \text{sech}(x+3)\exp(i2.3t) + 2\text{sech}(x)\tanh(x)\exp(i2.8t). \tag{14}$$

This example is demonstrated in [9] with the original DMD algorithm.

The individual spatiotemporal signals $f_1(x,t)$ and $f_2(x,t)$ are illustrated in Figure 2 (i)-(ii). The mixed signal $x(t) = f_1(x,t) + f_2(x,t)$ is illustrated in Figure 2 (iii). The two frequencies present are $\omega_1 = 2.3$ and $\omega_2 = 2.8$, which have distinct spatial structures.

Figure 1 shows the first 50 singular values of matrix X. Although the dynamics are constructed from a two-mode interaction, we need approximately 10 modes to produce an accurate reconstruction and characterize the dynamics.

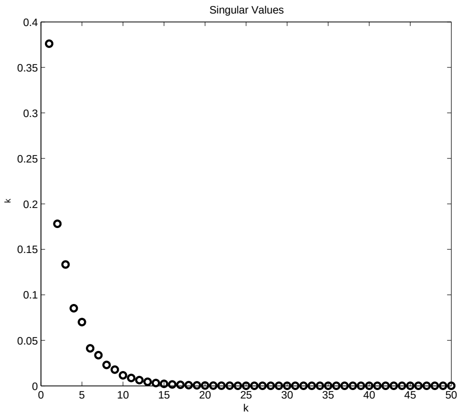

Figure 1: The singular value decay shows that at least rank-10 reconstruction is needed.

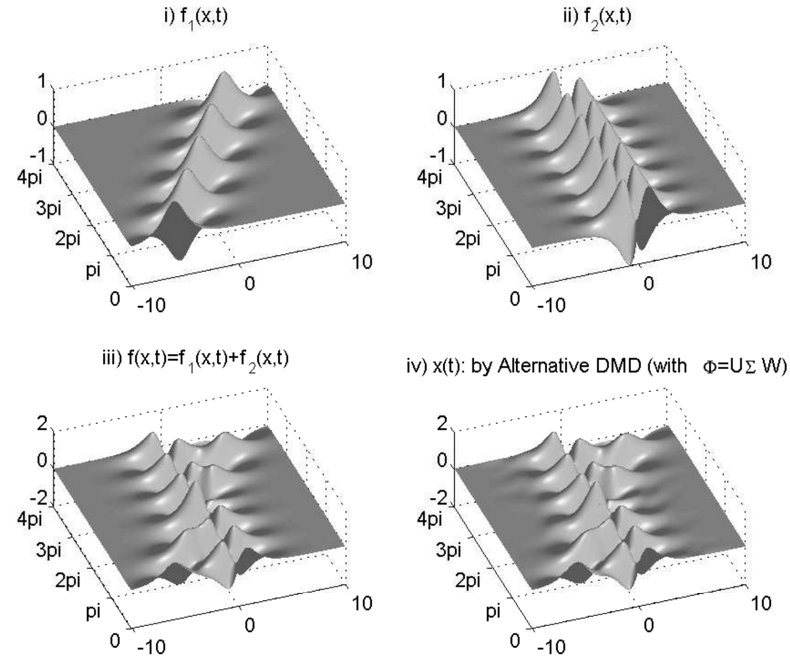

Figure 2: Spatiotemporal dynamics of two signals with translation (i) $\mathbf{f_1(x,t)}$ and (ii) $\mathbf{f_2(x,t)}$ of Example 1 that are mixed in (iii) $\mathbf{f_1(x,t)+f_2(x,t)}$. The *alternative exact DMD* is shown in (iv).

We perform a rank-10 approximate reconstruction (8) by using Algorithm 2, to illustrate the low-dimensional nature of the decomposition. The corresponding approximate reconstruction of $x(t)$ is shown in Figure 2 (iv). The reconstruction is almost perfect. The results show that alternative algorithms (*Algorithm 2* and *Algorithm 2*) produce the same result as the exact DMD procedure.

5 Conclusion

The purpose of this study was to explore a new algorithm for computing approximate DMD modes and eigenvalues. We demonstrate the performance of the presented algorithm with numerical results. From the obtained results we can conclude that the introduced approach gives identical results with those of the *exact DMD method*.

REFERENCES:

[1] Schmid, P.J. and Sesterhenn, J., Dynamic mode decomposition of numerical and experimental data. In 61st Annual Meeting of the APS Division of Fluid Dynamics. American Physical Society, (2008).

[2] Rowley, C. W., Mezić, I., Bagheri, S., Schlatter, P., and Henningson, D.S., Spectral analysis of nonlinear flows, J. Fluid Mech., **641** (2009), 115–127.

[3] Mezić, I., Spectral properties of dynamical systems, model reduction and decompositions, Nonlin. Dynam., **41** (1-3) (2005), 309–325.

[4] Schmid, P.J., Dynamic mode decomposition of numerical and experimental data, J. Fluid Mech., **656** (2010), 5–28.

[5] Schmid, P.J., Application of the dynamic mode decomposition to experimental data, Exp. Fluids, **50** (4) (2011), 1123–1130.

[6] Seena, A. and Sung, H.J., Dynamic mode decomposition of turbulent cavity ows for selfsustained oscillations, Int. J. Heat Fluid Fl., **32** (6) (2011), 1098–1110.

[7] Mezić, I., Analysis of fluid flows via spectral properties of the Koopman operator, Annu. Rev. Fluid Mech., **45** (2013), 357–378.

[8] Tu, J.H., Rowley, C. W., Luchtenburg, D.M., Brunton, S.L., Kutz, J.N., On dynamic mode decomposition: Theory and applications. Journal of Computational Dynamics, **1** (2) (2014), 391–421. doi: 10.3934/jcd.2014.1.391

[9] Kutz, J.N., Brunton, S.L., Brunton, B.W., and Proctor, J., Dynamic Mode Decomposition: Data-Driven Modeling of Complex Systems, SIAM 2016, ISBN 978-1-611-97449-2, (2016), 1–234.

[10] Chen, K. K., Tu, J.H., and Rowley, C. W., Variants of dynamic mode decomposition: Boundary condition, Koopman, and Fourier analyses. J. Nonlinear Sci., **22** (2012), 887–915.

[11] S. Bagheri, Koopman-mode decomposition of the cylinder wake, J. Fluid Mech., **726** (2013), 596–623, 2013.

[12] Grosek, J. , Kutz, J.N., Dynamic Mode Decomposition for Real-Time Background/Foreground Separation in Video, arXiv: 1404.7592.

[13] Proctor, J. L. and Eckhoff, P. A., Discovering dynamic patterns from infectious disease data using dynamic mode decomposition, International health, **7** (2015), 139–145.

[14] Berger, E., Sastuba, M., Vogt, D., Jung, B., and Amor, H. B., Estimation of perturbations in robotic behavior using dynamic mode decomposition, Journal of Advanced Robotics, **29** (2015), 331–343.

[15] Brunton, B. W., Johnson L. A., Ojemann J. G., and Kutz, J. N., Extracting spatial–temporal coherent patterns in large-scale neural recordings using dynamic mode decomposition, Journal of Neuroscience Methods, **258** (2016), 1–15.

[16] Mann, J. and Kutz, J. N., Dynamic mode decomposition for financial trading strategies, Quantitative Finance, (2016), 1–13.

[17] Ling-xiao Cui, Wen Long, 2016. "Trading strategy based on dynamic mode decomposition: Tested in Chinese stock market," Physica A: Statistical Mechanics and its Applications, Elsevier, **461** (C) (2016), 498–508.

[18] Kuttichira, D. P., Gopalakrishnan, E. A., Menon, V. K. and Soman, K. P., Stock price prediction using dynamic mode decomposition, 2017 International Conference on Advances in Computing, Communications and Informatics (ICACCI), Udupi, India, (2017), 55–60, doi: 10.1109/ICACCI.2017.8125816.

[19] Bai, Z., Kaiser, E., Proctor, J.L., Kutz, J.N., Brunton, S.L., Dynamic Mode Decomposition for CompressiveSystem Identification, AIAA Journal, **58** (2) (2020), 561–574.

[20] Nedzhibov, G., Dynamic Mode Decomposition: a new approach for computing the DMD modes and eigenvalues, Ann. Acad. Rom. Sci., Ser. Math. Appl. **14** (1-2) (2022), 5–16.