НУЛЕВИ МНОЖЕСТВА НА ХИПЕР-ПРОИЗВЕДЕНИЯ НА БЛАШКЕ И ИНТЕРПОЛАЦИОННИ РЕДИЦИ В ГОЛЕМИЯ КРЪГ

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ZERO SETS OF HYPER-BLASCHKE PRODUCTS AND INTERPOLATING SEQUENCES IN THE BIG DISK

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ABSTRACT: *Let G be the compact group of all characters of the additive group of rational numbers. We study the zero sets of hyper-Blaschke products in the big-disk Δ_G.*

KEYWORDS: *bounded hyper-analytic functions, zero sets, interpolating Blaschke products*

1. Introduction

Let Γ be a subgroup of the additive group of real numbers $\mathbb{R}$. We assume that Γ is equipped with the discrete topology. Denote by G the dual group of Γ, that is, G is the group of all continuous characters of Γ. Note that G is a compact abelian group with unit. By the Pontryagin duality theorem, the dual group $\widehat{G}$ of G is isomorphic to Γ and each character on G is of the type χ_p, $p \in \Gamma$, where $\chi_p(g) = g(p)$, $g \in G$. The open unit big-disk Δ_G over G we call the set $\Delta_G = [0,1) \times G / \{0\} \times G$. The points of Δ_G are denoted by $r.g$, $0 \le r < 1$ with the understanding that all points of type $0.g$ are identified into a single point, $\{*\}$, the origin (or the center) of big-disk Δ_G. For any $p \in \Gamma_+ = \Gamma \cap [0,\infty)$ the character χ_p on G is extendable up to the closed big disk $\overline{\Delta}_G = [0,1] \times G / \{0\} \times G$ in the following way: $\tilde{\chi}_p(r.g) = r^p.\chi(g)$ for $p \ne 0$ and $r \ne 0$, $\tilde{\chi}_p(*) = 0$ for any $p \ne 0$ and $\tilde{\chi}_0 \equiv 1$ on $\overline{\Delta}_G$. Each function $\tilde{\chi}_p$; $p \in \Gamma_+ \setminus \{0\}$, projects the closed big-disk $\overline{\Delta}_G$ onto the closed unit disk $\overline{\Delta}$ and the open big disk Δ_G onto the open unit disk Δ in the complex plane.

The uniform closure A_G of finite linear combinations of functions $\tilde{\chi}_p$, with complex coefficients, i.e. of generalized polynomials, is the big-disk algebra on $\overline{\Delta}_G$. A_G is a uniform algebra on $\overline{\Delta}_G$ and its elements are called generalized-analytic functions in the sense of R. Arens and A. Singer. The maximal ideal space (spectrum) $M(A_G)$ of the big-disk algebra is the closed unit big-disk $\overline{\Delta}_G$ ([1]).

Note that if Γ is the additive group of integers $\mathbb{Z}$, then its dual, $\widehat{\Gamma} = \widehat{\mathbb{Z}}$, is the unit circle $T = \partial\Delta$ in the complex plane, the open big-disk $\Delta_G = \Delta_T$ is the open unit disk Δ in the complex plane, and the corresponding big-disk algebra, $A_T = A(\overline{\Delta})$, the classical disk algebra. In this paper we consider the case $\Gamma = \mathbb{Q}$ - the additive group of rational numbers and $G = \widehat{\mathbb{Q}}$.

Let Γ be the group of rational numbers $\mathbb{Q}$ and $G=\widehat{\mathbb{Q}}$. A function f on the open unit big-disk is said to be hyper-analytic on Δ_G if f can be approximated uniformly on Δ_G by functions of type $h\circ\tilde{\chi}_{1/n}$, where $n\in\mathbb{Z}_+=\mathbb{Z}\cap(0,\infty)$ and h is analytic on the open unit disk Δ.

The algebra of all bounded hyper-analytic functions on Δ_G is denoted by H_G^∞. This algebra were introduced by T. Tonev $([2],[3])$. Under the sup-norm on Δ_G, H_G^∞ is a commutative Banach algebra with unit. As customary, we identify the functions $f\in H_G^\infty$ with their Gelfand transforms $\hat{f}\in\widehat{H_G^\infty}$ defined by $\hat{f}(\varphi)=\varphi(f)$, where φ runs in the spectrum $M\left(H_G^\infty\right)$.

In $[4]$ S. Grigorian and T. Tonev study Blaschke inductive limit algebras $A(b)$, defined as inductive limits of disk algebras $A(T)$ linked by a sequence $b=\{B_k\}_1^\infty$ of finite Blaschke products. It is shown that a big - disk algebra A_G over a group G with ordered dual $\Gamma=\widehat{G}\subset\mathbb{R}$ is a Blaschke inductive limit algebra if and only if $\Gamma=\widehat{G}\subset\mathbb{Q}$. They consider also inductive limits $H^\infty(I)$ of classical algebras H^∞ of bounded analytic functions on the open unit disk Δ, linked by a sequence $I=\{I_k\}_1^\infty$ of inner functions, and prove a version of the corona theorem with estimates for it. The algebra $H^\infty(I)$ generalizes the algebra H_G^∞ of bounded hyper-analytic functions on an open big – disk.

The pseudohyperbolic distance between two points φ and ψ in the spectrum $M\left(H^\infty\right)$ is defined by $\rho_{H^\infty}(\varphi,\psi)=\sup\left\{|h(\varphi)|:h\in\mathrm{ball}\left(H^\infty\right),h(\psi)=0\right\}$, where $\mathrm{ball}\left(H^\infty\right)$ stands for the closed unit ball of H^∞. By Schwarz-Pick's lemma $\rho_{H^\infty}(z,w)=|z-w|/|1-\bar{z}w|$ for z and w in Δ $([5])$. As in H^∞ the H_G^∞ - pseudohyperbolic distance in $M\left(H_G^\infty\right)$ is given by

$$\rho_{H_G^\infty}(\varphi,\psi)=\sup\left\{|h(\varphi)|:h\in\mathrm{ball}\left(H_G^\infty\right),h(\psi)=0\right\},$$

where φ and ψ belong to $M\left(H_G^\infty\right)$**.** The link between H_G^∞ - pseudohyperbolic distance in Δ_G and H^∞ - pseudohyperbolic distance in Δ is as follows:

$$\rho_{H_G^\infty}(r_1.g_1,r_2.g_2)=\sup_{m\in\mathbb{Z}_+}\rho_{H^\infty}\left(\tilde{\chi}_{1/m}(r_1.g_1),\tilde{\chi}_{1/m}(r_2.g_2)\right).$$

This equality and other relationships between Gleason parts in $M\left(H_G^\infty\right)$ and $M\left(H^\infty\right)$ are proven in $([6])$.

A sequence $\{\varphi_n\}_1^\infty$ in $M\left(H^\infty\right)$ is called interpolating if for every bounded sequence $\{a_n\}_1^\infty$ of complex numbers there is a function $f\in H^\infty$ such that $f(\varphi_n)=a_n$ for all n. A sequence $\{\varphi_n\}_n$ in $M\left(H^\infty\right)$ is said to be discrete if there exists a sequence of open sets $\{U_n\}_1^\infty$

with $\varphi_n \in U_n$ for every n, whose closures are pairwise disjoint. Every interpolating sequence is discrete.

An interpolating sequence $\{z_n\}_1^\infty$ in Δ is characterized by Carleson $[5]$ as follows

$$\inf_j \prod_{n:n\neq j} \rho_{H^\infty}\left(z_j, z_n\right) > 0 .$$

For a sequence $\{z_n\}_1^\infty$ in Δ with $\sum_n \left(1-|z_n|\right) < \infty$, the function

$$B(z) = \prod_n \frac{-\overline{z}_n}{|z_n|}\frac{z-z_n}{1-\overline{z}_n z}, \; z \in \Delta,$$

is called a Blaschke product with zeros $\{z_n\}_1^\infty$. If $\{z_n\}_1^\infty$ is an interpolating sequence, then $B(z)$ is also called interpolating. Each Blaschke product B is an inner function, i.e. $\left|B\left(e^{i\theta}\right)\right| = 1$ almost everywhere on T.

In this paper we study the zero sets of interpolation hyper – Blaschke products, i.e. the functions of the type $b_m = b \circ \tilde{\chi}_{1/m}$, where b is a Blaschke product on Δ and $m \in \mathbb{Z}_+$. Also we prove certain properties to a class of interpolating sequences in Δ_G.

2. Zero sets of hyper-Blaschke product and interpolating sequences in Δ_G

Given an $m \in \mathbb{Z}_+$ the set $H_{1/m}^\infty = \left\{h \circ \tilde{\chi}_{1/m} : h \in H^\infty\right\}$ is a subalgebra of H_G^∞. It is easy to see that $H_{1/n}^\infty \subset H_{1/m}^\infty$ whenever $m = kn$ for some $k \in \mathbb{Z}_+$. The map $h \to h \circ \tilde{\chi}_{1/m}$ is an isometric algebra isomorphism between H^∞ and $H_{1/m}^\infty$. By definition the set $\bigcup_m H_{1/m}^\infty$ is dense in H_G^∞, i.e. the closure $\overline{\left\{H_{1/m}^\infty\right\}_1^\infty}$ coincides with H_G^∞.

For an arbitrary function $f \in H^\infty$ and $m \in \mathbb{Z}_+$ we denote the zero set of f in Δ with $N(f)$ and with $N(f_m)$ - the zero set of $f_m = f \circ \tilde{\chi}_{1/m} \in H_{1/m}^\infty$ in Δ_G i.e. $N(f_m) = \left\{\tilde{\chi}_{1/m}^{-1}\left(N(f)\right)\right\}$.

Let b be a Blaschke product on the open unit disk Δ and $m \in \mathbb{Z}_+$. The function of the type $b_m = b \circ \tilde{\chi}_{1/m}$ we call hyper – Blaschke product on the open big disk Δ_G. The hyper – Blaschke product b_m is an inner function in $H_{1/m}^\infty \subset H_G^\infty$ with zero set $N(b_m) = \left\{\tilde{\chi}_{1/m}^{-1}(\omega_n)\right\}_{n=1}^\infty \subset \Delta_G$, where $\{\omega_n\}_1^\infty \subset \Delta$ are all zeros of b. If b is interpolating, then b_m also called interpolating hyper – Blaschke product. Since for $t \in \mathbb{Z}_+$ we have

$$\tilde{\chi}_{1/tm}\left(r^t.g^t\right) = \left(r^t\right)^{1/tm}.g^t\left(1/tm\right) = r^{1/m}.g\left(1/m\right) = \tilde{\chi}_{1/m}\left(r.g\right),$$

then $N(b_{tm}) = \left\{r^t.g^t : r.g \in N(b_m)\right\}$.

For each real number s, the character e_s is defined by $e_s(p)=e^{isp}$ for $p\in\mathbb{Q}$. The induced mapping $s\to e_s$ is an isomorphism of the real line $\mathbb{R}$ into a dense subgroup of G $([1])$. If $r.g\neq *$ is in Δ_G, this isomorphism can be extended to the upper half-plane $\mathbb{C}_+=\{z\in\mathbb{C}:\operatorname{Im} z>0\}$ as follows: $J_g(s+it)=e^{-t}.ge_s$. The image $J_g(\mathbb{C}_+)$ contains the point $r.g$ and is dense in Δ_G. This image J_g is called standard embedding.

Example. Let α be a point in $\Delta\setminus\{0\}$, $f(z)=z-\alpha$ and $f_m=f\circ\tilde{\chi}_{1/m}=\tilde{\chi}_{1/m}-\alpha$, $m\in\mathbb{Z}_+$. To study the structure of zero set $N(f_m)$ we can use standard embedding J_g of the upper half-plane $\mathbb{C}_+$ in the open big disk Δ_G. If $\alpha=|\alpha|.e^{i\varphi}$, where φ is an arbitrary argument of α, then the equation $\tilde{\chi}_{1/m}(r.e_s)-\alpha=0\Leftrightarrow r^{1/m}.e^{is/m}=|\alpha|.e^{i\varphi}$ has on $J_{e_0}(\mathbb{C}_+)$ countable many solutions:

$$Z_{e_0}^{1/m}=\left\{|\alpha|^m.e_{s_k}\right\}_{k\in\mathbb{Z}}\subset|\alpha|^m\times G,$$

$s_k=(\varphi+2k\pi)m$. Let $g_0\in G$ and $g_0\neq e_s$ for every $s\in\mathbb{R}$. Then the zero set of f_m in the "upper half-plane" $J_{g_0}(\mathbb{C}_+)=g_0.J_{e_0}(\mathbb{C}_+)$ in Δ_G is the set $Z_{g_0}^{1/m}=\left\{|\alpha|^m.g_0e_{s_k-\theta m}\right\}_{k\in\mathbb{Z}}\subset|\alpha|^m\times G$, where θ is a fixed value of the argument of $g_0(1/m)$. Since the multiplication with $g_0e_{-\theta m}$ is a homeomorphism of G on G, then $Z_{e_0}^{1/m}$ and $Z_{g_0}^{1/m}$ are homeomorphic. Hence every two zero sets $Z_{g_1}^{1/m}\subset J_{g_1}(\mathbb{C}_+)$ and $Z_{g_2}^{1/m}\subset J_{g_2}(\mathbb{C}_+)$ of f_m are homeomorphic. So for zero set of f_m we obtain $N(f_m)=\bigcup\limits_{g\in G}Z_g^{1/m}$.

Let $|\alpha|^m.g_0e_{s_k-\theta m}$ be an arbitrary point of $N(f_m)$ and we denote by $W_\varepsilon=U_\varepsilon\times V_\varepsilon$, where

$U_\varepsilon=\left\{r:\left|r-|\alpha|^m\right|<\varepsilon\right\}$ and $V_\varepsilon=\left\{g\in G:\left|g(p_j)-\left(g_0e_{s_k-\theta m}\right)(p_j)\right|<\varepsilon, p_j\in\mathbb{Q}_+, j=1,2,...,n\right\}$,

it's a basic neighborhood.

If $p_j=\alpha_j/\beta_j$ and $t=\beta_1.\beta_2...\beta_n$, then $\left[g_0(1/mt)\right]^t=g_0(1/m)=e^{i\theta}$ and there exists $l\in\mathbb{Z}_+$ such that $g_0(1/mt)=e^{i\theta/t}.e^{i2\pi l/t}$. For $r_j=p_j.tm=\alpha_j.\beta_1...\beta_{j-1}.\beta_{j+1}...\beta_n.m$ we have:

$$g_0(p_j)=g_0(\alpha_j/\beta_j)=\left[g_0(1/mt)\right]^{\alpha_j.\beta_1...\beta_{j-1}.\beta_{j+1}...\beta_n.m}=\left[e^{i\theta/t}\right]^{r_j}.\left[e^{i2\pi l/t}\right]^{r_j}=e^{i\theta p_j.m}.e^{i2\pi lp_j.m}$$

and

$$\left(g_0.e_{s_k-\theta m}\right)(p_j)=g_0(p_j).e^{is_kp_j}.e^{-i\theta mp_j}=e^{i\theta p_jm}.e^{i2\pi lp_j.m}e^{is_kp_j}.e^{-i\theta mp_j}=e^{is_kp_j}.e^{i2\pi lp_j.m}=e_{s_k+2\pi lm}(p_j),$$

$j=1,2,...,n$, i.e. $|\alpha|^m.e_{s_k+2\pi lm}=|\alpha|^m.e_{s_{k+l}}\in W_\varepsilon\cap Z_{e_0}^{1/m}$.
Therefore, $Z_{e_0}^{1/m}$ is dense in $N(f_m)$.

Theorem 2.1. Let $b_m = b \circ \tilde{\chi}_{1/m}$ for $m \in \mathbb{Z}_+$ be a hyper – Blaschke product on the open big disk Δ_G. Then for every $t \in \mathbb{Z}_+$ there exist Blaschke products $b^{(1)}, b^{(2)}, ..., b^{(t)}$ on the open unit disk Δ such that:

(1) For every $j = 1 \div t$ is fulfilled $\omega\left(N\left(b^{(j)}\right)\right) = N(b)$, where $\omega(z) = z^t$. The functions $b \circ \omega$ and $B = b^{(1)} b^{(2)} ... . b^{(t)}$ have the same zeros in Δ which is a single.

(2) $N\left(b_{tm}^{(j)}\right) \cap N\left(b_{tm}^{(i)}\right) = \varnothing$ for $j \neq i$ and $N(B_{tm}) = \bigcup_{j=1}^{t} N\left(b_{tm}^{(j)}\right) = N(b_m)$.

(3) b_m has the form $b_m = b_{tm}^{(1)} . b_{tm}^{(2)} ... b_{tm}^{(t)}$ in Δ_G.

Proof: Let b be a Blaschke product with zeros $\{\omega_n\}_1^\infty$ in unit disk Δ, $\omega_n \neq \omega_k$ for $n \neq k$, $N(b_m) = \left\{\tilde{\chi}_{1/m}^{-1}(\omega_n)\right\}_{n=1}^\infty \subset \Delta_G$ and $t \in \mathbb{Z}_+$. Since for $r.g \in \Delta_G$ and $z = \tilde{\chi}_{1/tm}(r.g) \in \Delta$ we have $z^t = \left[\tilde{\chi}_{1/tm}(r.g)\right]^t = \tilde{\chi}_{1/m}(r.g)$, then $\tilde{\chi}_{1/tm}(N(b_m))$ is the set of all points z in Δ, such that $z^t = \omega_n$ for some n. For fixed n these points are a finite number and we obtain that the $\tilde{\chi}_{1/tm}(N(b_m))$ is a countable set.

If $\{z_s\}_1^\infty = \tilde{\chi}_{1/tm}(N(b_m))$ we can consider, that $z_1, z_2, ..., z_t$ are reflected with $\omega(z) = z^t$ on ω_1; $z_{t+1}, z_{t+2}, ..., z_{t+t}$ - on ω_2 etc., i.e. $\left(z_{nt+j}\right)^t = \omega_{n+1}$ for $n = 0,1,2,...$ and $j = 1,2,...,t$. It is also clear that this sequence is composed of different points. Consider the following finite parts of the sequence $\{z_s\}_1^\infty$:

$$a_{11} = z_1, a_{21} = z_{t+1}, ..., a_{n1} = z_{(n-1)t+1}, ...$$
$$a_{12} = z_2, a_{22} = z_{t+2}, ..., a_{n2} = z_{(n-1)t+2}, ...$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{1t} = z_t, a_{2t} = z_{t+t}, ..., a_{nt} = z_{(n-1)t+t}, ...$$

Since $\sum_{n=1}^{\infty}(1 - |\omega_n|) < \infty$ and $|\omega_n| = \left|\left(a_{nj}\right)^t\right| < |a_{nj}|$, then $\sum_{n=1}^{\infty}\left(1 - |a_{nj}|\right) < \infty$ for every $j = 1,2,...,t$. We denote by $b^{(1)}, b^{(2)}, ..., b^{(t)}$ the Blaschke products with zero sets, respectively $\{a_{n1}\}_1^\infty, \{a_{n2}\}_1^\infty, ..., \{a_{nt}\}_1^\infty$.

(1) Obviously $\omega\left(N\left(b^{(j)}\right)\right) = N(b)$, where $\omega(z) = z^t$. Since each of the sequences $\{z_s\}_1^\infty$, $\{a_{nj}\}_{n=1}^\infty$, $j = 1,2,...,t$, and $\{\omega_n\}_1^\infty$ are composed of different points, then the functions $b \circ \omega$ and $B = b^{(1)} b^{(2)} ... . b^{(t)}$ have the same zeros in Δ, which are single. Of course B it is also a Blaschke product with zeros $\{z_s\}_1^\infty$.

(2) If $j \neq i$, then $N\left(b_{tm}^{(j)}\right) \cap N\left(b_{tm}^{(i)}\right) = \left\{\tilde{\chi}_{1/tm}^{-1}\left(a_{nj}\right)\right\}_{n=1}^{\infty} \cap \left\{\tilde{\chi}_{1/tm}^{-1}\left(a_{ni}\right)\right\}_{n=1}^{\infty} = \varnothing$, because then $\left\{a_{nj}\right\}_{n=1}^{\infty} \cap \left\{a_{ni}\right\}_{n=1}^{\infty} = \varnothing$. The quality $N\left(B_{tm}\right) = \bigcup_{j=1}^{t} N\left(b_{tm}^{(j)}\right) = N\left(b_m\right)$ followed by the presentation $B = b^{(1)}b^{(2)}....b^{(t)}$ and $\left\{z_s\right\}_1^{\infty} = \left\{a_{nj}\right\}_{n=1,\, j=1}^{\infty\ \ ,t} = \chi_{1/tm}\left(N\left(b_m\right)\right)$.

(3) By (1) the functions $b \circ \omega$ и $B = b^{(1)}b^{(2)}....b^{(t)}$ have the same zeros $\left\{z_s\right\}_1^{\infty}$ in Δ, which is a single. Then the functions $\left(b \circ \omega\right)/B$ и $B/\left(b \circ \omega\right)$ are analytical in Δ and their modules does not exceed unit. By the maximum principle for holomorphic functions we have that $\left(b \circ \omega\right)/B \equiv 1$ in Δ. Therefore, $\left(b \circ \omega\right)\left(z\right) = B\left(z\right)$ for every $z \in \Delta$ and we obtain:

$$b_m\left(r.g\right) = \left(b \circ \tilde{\chi}_{1/m}\right)\left(r.g\right) = \left(b \circ \omega\right) \circ \tilde{\chi}_{1/tm}\left(r.g\right) = b_{tm}^{(1)}\left(r.g\right).b_{tm}^{(2)}\left(r.g\right)...b_{tm}^{(t)}\left(r.g\right)$$

for every $r.g \in \Delta_G$.

Note that if b is an interpolating Blaschke product on Δ, then $b^{(1)}, b^{(2)}, ..., b^{(t)}$ are interpolating also. Indeed, then for each $j = 1, 2, ..., t$ the sequence $\left\{a_{nj}\right\}_{n=1}^{\infty}$ is interpolating, because $\left\{\omega_n\right\}_1^{\infty}$ is an interpolating sequence and $\omega\left(a_{nj}\right) = \omega_n$ for every n.

A sequence $\left\{r_k.g_k\right\}_1^{\infty} \subset \Delta_G$ is called interpolating for $H_{1/n}^{\infty} \subset H_G^{\infty}$, if for every sequence of complex numbers $\left\{a_k\right\}_1^{\infty} \subset l^{\infty}$ there is a function $h \in H^{\infty}$ such that $\left(h \circ \tilde{\chi}_{1/n}\right)\left(r_k.g_k\right) = a_k$ for all k. It is clear that the sequence $\left\{r_k.g_k\right\}_1^{\infty}$ is interpolating for $H_{1/n}^{\infty}$, if and only if the sequence $\left\{\tilde{\chi}_{1/n}\left(r_k.g_k\right)\right\}_1^{\infty} \subset \Delta$ is interpolating for H^{∞}. It is characterized by Carleson with the inequality

$$\inf_j \prod_{k:k \neq j} \rho_{H^{\infty}}\left(\tilde{\chi}_{1/n}\left(r_k.g_k\right), \tilde{\chi}_{1/n}\left(r_j.g_j\right)\right) \geq \delta > 0\,.$$

Also, if a sequence is interpolating for $H_{1/n}^{\infty}$, she is interpolating and for $H_{1/mn}^{\infty} \supset H_{1/n}^{\infty}$, i.e.

$$\inf_j \prod_{k:k \neq j} \rho_{H^{\infty}}\left(\tilde{\chi}_{1/mn}\left(r_k.g_k\right), \tilde{\chi}_{1/mn}\left(r_j.g_j\right)\right) \geq \delta > 0\,,$$

for every $m \in \mathbb{Z}_+$.

Let $\left\{r_k.g_k\right\}_1^{\infty}$ be an interpolating sequence for $H_{1/m}^{\infty}$, $\omega_k = \tilde{\chi}_{1/m}\left(r_k.g_k\right)$ for $k \in \mathbb{N}$ and b be an interpolating Blaschke product with zeros $\left\{\omega_k\right\}_1^{\infty}$. Then the zero set of interpolating hyper – Blaschke product $b_m = b \circ \tilde{\chi}_{1/m}$ has the form $N\left(b_m\right) = \bigcup_{k=1}^{\infty} N_k\left(b_m\right)$, where $N_k\left(b_m\right)$ is the zero set of $\tilde{\chi}_{1/m} - \omega_k$ (see example). The sequence $\left\{r_k.g_k\right\}_1^{\infty}$ is a small part of this set. Not every sequence of points in $N\left(b_m\right)$ is interpolating. But every sequence $\left\{r_k.g_k\right\}_1^{\infty} \subset N\left(b_m\right)$, such that $r_k \neq r_j$ for

$k \neq j$ is interpolating for $H^{\infty}_{1/m}$. Indeed, then $\left\{\tilde{\chi}_{1/m}\left(r_k.g_k\right)\right\}_1^{\infty}$ is a subsequence of a sequence $\left\{\omega_k\right\}_1^{\infty}$ and therefore is interpolating for H^{∞}.

Proposition 2.2. Let $\left\{r_k.g_k\right\}_1^{\infty}$ be an interpolating sequence for $H^{\infty}_{1/n}$. Then:

(1) $\left\{r_k.g_k\right\}_1^{\infty}$ is a separated sequence, i.e. there exists $\alpha > 0$ such that $\rho_{H_G^{\infty}}\left(r_k.g_k, r_j.g_j\right) \geq \alpha$ for $j \neq k$.

(2) $\left\{r_k.g_k\right\}_1^{\infty}$ is a discrete.

(3) $\left\{r_k.g_k\right\}_1^{\infty}$ satisfies the Carleson condition in Δ_G, i.e.

$$\inf_j \prod_{k:k\neq j} \rho_{H_G^{\infty}}\left(r_k.g_k, r_j.g_j\right) \geq \delta > 0 .$$

(4) Every sequence $\left\{z_k\right\}_1^{\infty} \subset \mathbb{C}_+$ such that $\left(\tilde{\chi}_{1/n} \circ J_{g_0}\right)\left(z_k\right) = \tilde{\chi}_{1/n}\left(r_k.g_k\right)$ for some $g_0 \in G$ and for every k, is interpolating for $H^{\infty}\left(\mathbb{C}_+\right)$.

Proof: (1) Since the sequence $\left\{\tilde{\chi}_{1/n}\left(r_k.g_k\right)\right\}_1^{\infty} \subset \Delta$ is interpolating for H^{∞}, then there exists $\alpha > 0$ such that

$$\rho_{H_G^{\infty}}\left(r_k.g_k, r_j.g_j\right) = \sup_{m\in\mathbb{Z}_+} \rho_{H^{\infty}}\left(\tilde{\chi}_{1/m}\left(r_k.g_k\right), \tilde{\chi}_{1/m}\left(r_j.g_j\right)\right) \geq \rho_{H^{\infty}}\left(\tilde{\chi}_{1/n}\left(r_k.g_k\right), \tilde{\chi}_{1/n}\left(r_j.g_j\right)\right) \geq \alpha .$$

(2) $\left\{r_k.g_k\right\}_1^{\infty}$ is discrete, because $\left\{\tilde{\chi}_{1/n}\left(r_k.g_k\right)\right\}_1^{\infty} \subset \Delta$ is a discrete sequence and $\tilde{\chi}_{1/n}$ is continuous in Δ_G.

(3) Followed by the inequality

$$\rho_{H_G^{\infty}}\left(r_k.g_k, r_j.g_j\right) \geq \rho_{H^{\infty}}\left(\tilde{\chi}_{1/n}\left(r_k.g_k\right), \tilde{\chi}_{1/n}\left(r_j.g_j\right)\right)$$

and that the sequence $\left\{\tilde{\chi}_{1/n}\left(r_k.g_k\right)\right\}_1^{\infty} \subset \Delta$ is interpolating for H^{∞}.

(4) If $z = x + iy \in \mathbb{C}_+$, then the function:

$$\left(\tilde{\chi}_{1/n} \circ J_{g_o}\right)(z) = \tilde{\chi}_{1/n}\left(e^{-y}.g_0 e_x\right) = e^{-y/n}.g_0(1/n).e^{ix/n} = e^{iz/n}.g_0(1/n)$$

is analytic in $\mathbb{C}_+$ and $\left(\tilde{\chi}_{1/n} \circ J_{g_o}\right)\left(\mathbb{C}_+\right) \subset \Delta$. Since $\left\{\tilde{\chi}_{1/n}\left(r_k.g_k\right)\right\}_1^{\infty} \subset \Delta$ is interpolating for H^{∞} and $\left(\tilde{\chi}_{1/n} \circ J_{g_0}\right)\left(z_k\right) = \tilde{\chi}_{1/n}\left(r_k.g_k\right)$, then $\left\{z_k\right\}_1^{\infty}$ is an interpolating sequence for $H^{\infty}\left(\mathbb{C}_+\right)$.

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