A VISUALIZATION AND A SHAPE CHARACTERISATION OF A CLASS OF CYLINDRICAL HELICES *

GEORGI H. GEORGIEV, RADOSTINA P. ENCHEVA, CVETELINA L. DINKOVA

ABSTRACT: *In this paper we investigate parameterizations of cylindrical helices generated by unit speed plane curves. We also examine properties of their focal curves. Illustrative examples of cylindrical helices with periodical curvature and torsion are presented.*

KEYWORDS: *Cylindrical Helix, Curvature, Torsion, Focal Curve*

1 Introduction

Constructing a new curve from a given regular curve is an important task in the classical differential geometry. In this paper we examine cylindrical helices generated by unit speed plane curves of class C^3 and their focal curves. Any unit speed plane curve is uniquely determined up to Euclidean motion of $\mathbb{E}^2$ by its curvature function K. Every plane curve with a non-zero curvature is fully determined up to direct similarity by another function $\widetilde{K}$ called a shape curvature. For any regular parameterized curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(t) : I \longrightarrow \mathbb{E}^2$ the shape curvature function $\widetilde{K}(t)$ can be expressed by $\widetilde{K}(t) = \dfrac{d}{dt}\left(\dfrac{1}{K(t)}\right) / \left\|\dfrac{d\boldsymbol{\alpha}}{dt}\right\|$. Analogously, any unit speed space curve is uniquely determined up to Euclidean motion of $\mathbb{E}^3$ by its curvature $\kappa_1 \geq 0$ and its torsion κ_2. If $\kappa_1 > 0$ this curve is fully determined up to direct similarity by other functions $\widetilde{\kappa}_1$ and $\widetilde{\kappa}_2$ called a shape curvature and a shape torsion, respectively. Any regular parameterized space curve $\boldsymbol{\gamma} = \boldsymbol{\gamma}(t) : I \longrightarrow \mathbb{E}^3$ has a shape curvature and a shape torsion defined by functions

$$(1) \qquad \widetilde{\kappa}_1(t) = \frac{\frac{d}{dt}\left(\frac{1}{\kappa_1(t)}\right)}{\left\|\frac{d\boldsymbol{\gamma}(t)}{dt}\right\|} \quad \text{and} \quad \widetilde{\kappa}_2(t) = \frac{\kappa_2(t)}{\kappa_1(t)}$$

The geometry of curves with respect to an orientation-preserving Euclidean motion is completely presented in [6]. For instance, one important theorem proved in this book is

Theorem 1.1. *[6, p.137](Fundamental Theorem of Plane Curves) A unit-speed curve $\boldsymbol{\alpha} : I \to \mathbb{E}^2$ whose curvature is given piecewise-continuous function $K : I \to \mathbb{R}$ is parameterized by*

$$(2) \qquad \begin{cases} \boldsymbol{\alpha}(s) = \left(\int \cos\theta(s)ds + d_1, \int \sin\theta(s)ds + d_2\right), \\ \theta(s) = \int K(s)ds + \theta_0, \end{cases}$$

where d_1, d_2, θ_0 are constants of integration.

Elements of the theory of curves related to direct similarities are given in [2], [3] and [4]. There are special space curves called cylindrical helices with the property $\dfrac{\kappa_2(t)}{\kappa_1(t)} = const$. These curves are studied in [7] and [8]. There exists a well-known construction for obtaining a new space curve from a given space curve.

*Partially supported by Scientific Research Grant RD–08–111/05.02.2018 of Shumen University.

Definition 1.1. *[6, p.241] A **focal curve** (or an evolute) of a regular C^3-space curve $\boldsymbol{\gamma}: (a,b) \to \mathbb{E}^3$ with a nonzero curvature and a torsion is the curve given by*

$$\mathbf{C}_\gamma(t) = \boldsymbol{\gamma}(t) + c_1(t)\mathbf{n_1}(t) + c_2(t)\mathbf{n_2}(t), \tag{3}$$

where $\mathbf{n_1}$ is a principal unit normal vector field of $\boldsymbol{\gamma}$, $\mathbf{n_2}$ is a binormal unit vector field of $\boldsymbol{\gamma}$. The coefficients $c_1(t)$ and $c_2(t)$ are smooth functions called focal curvatures of $\boldsymbol{\gamma}$, given by

$$c_1(t) = \frac{1}{\kappa_1(t)}, \quad c_2(t) = -\frac{\frac{d}{dt}\kappa_1(t)}{\| \frac{d\boldsymbol{\gamma}(t)}{dt} \| \kappa_1(t)^2\kappa_2(t)} = \frac{\frac{dc_1(t)}{dt}}{\| \frac{d\boldsymbol{\gamma}(t)}{dt} \| \kappa_2(t)}, \tag{4}$$

where $\kappa_1(t)$ and $\kappa_2(t)$ are the curvature and the torsion of $\boldsymbol{\gamma}$.

In this paper we describe a construction which is divided into three steps. First, we obtain a cylindrical helix from a given unit speed curve in the *xy*-plane. Second, we determine the focal curve of the obtained cylindrical helix. Third, we study the orthogonal projection of the focal curve on the *xy*-plane. This scheme applied to plane curves with periodic signed curvature is presented.

2 Parameterizations of cylindrical helices

Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, $s \in I \subseteq \mathbb{R}$ be a unit speed regular C^3-curve in the Euclidean plane $\mathbb{E}^2 \equiv O\vec{\mathbf{e}}_1\vec{\mathbf{e}}_2$ with a vector parametric equation $\boldsymbol{\alpha}(s) = (x(s), y(s), 0)$ parameterized by an arc-length parameter $s \in I$. The signed curvature of $\boldsymbol{\alpha}$ is

$$K =< \boldsymbol{\alpha}'', J\boldsymbol{\alpha}' >= x'(s).y''(s) - x''(s).y'(s), \tag{5}$$

where J is the complex structure of $\mathbb{R}^2$ and "$< \cdot, \cdot >$" denotes the scalar (or dot) product of two vectors. We define a new space curve $\boldsymbol{\gamma} = \boldsymbol{\gamma}(s)$, $s \in I \subseteq \mathbb{R}$ in the Euclidean space $\mathbb{E}^3 \equiv O\vec{\mathbf{e}}_1\vec{\mathbf{e}}_2\vec{\mathbf{e}}_3$ with a parametric representation

$$\boldsymbol{\gamma}(s) = (x(s), y(s), as+b) = \boldsymbol{\alpha}(s) + (as+b)\vec{\mathbf{e}}_3, \ a,b = const \tag{6}$$

This curve lies on the generalized cylinder S_α with a base curve $\boldsymbol{\alpha}(s)$ and rulings parallel to *z*-axis. The curvature and the torsion of $\boldsymbol{\gamma}$ are expressed by the next lemma.

Lemma 2.1. *[5, p.5640] Let $\boldsymbol{\gamma}$ be a space curve with a parametrization* (6). *Then the Euclidean curvature κ_1 and the Euclidean torsion κ_2 of $\boldsymbol{\gamma}$ are given by*

$$\kappa_1 = \frac{|K|}{1+a^2}, \quad \kappa_2 = \frac{a.K}{1+a^2}, \tag{7}$$

where K is the signed curvature of $\boldsymbol{\alpha}$.

Observe that for the curve $\boldsymbol{\gamma}$ the condition characterizing cylindrical helixes $\kappa_2/\kappa_1 = const$ holds. More precisely, $\boldsymbol{\gamma}$ is a cylindrical helix with a constant curvature ratio $\kappa_2/\kappa_1 = \varepsilon a$, where $\varepsilon = sign(K)$. In what follows we will call ***$\boldsymbol{\gamma}$ a cylindrical helix generated by the plane curve $\boldsymbol{\alpha}$***. Moreover, this cylindrical helix is a geodesic curve on S_α. Analogous condition for the focal curvatures c_1 and c_2 of $\boldsymbol{\gamma}$ can be written in the form $c_1.c_1' - \varepsilon a\sqrt{1+a^2}c_2 \equiv 0$. The next statement gives a relation between focal curvatures of $\boldsymbol{\gamma}$ and the signed curvature of $\boldsymbol{\alpha}$.

Theorem 2.2. *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, $s \in I$ be a unit-speed C^3-plane curve in $\mathbb{E}^2$ with a signed curvature $K \neq 0$. Let $\boldsymbol{\gamma}(s)$ be a space curve given by* (6). *If c_1 and c_2 are the focal curvatures of $\boldsymbol{\gamma}$, then*

$$c_1 = \frac{1+a^2}{|K|} \text{ and } c_2 = -\frac{\sqrt{1+a^2}^{\,3}.K'}{a.|K|^3}, \tag{8}$$

where a is the constant slope of $\boldsymbol{\gamma}$ with respect to $\vec{\mathbf{e}}_3$.

Proof. From (4) and (7) we get

$$c_1(s) = \frac{1}{\kappa_1} = \frac{1+a^2}{|K|} \quad \text{and} \quad c_2(s) = -\frac{\frac{d}{ds}\kappa_1(s)}{\| \frac{d\boldsymbol{\gamma}(s)}{ds} \| \, \kappa_1(s)^2 \kappa_2(s)} = -\frac{\sqrt{1+a^2}^3.K'}{a.|K|^3}.$$

□

Now, we give a useful relation between the shape curvature and the shape torsion of the cylindrical helix and the shape curvature of the generating plane curve.

Corollary 2.2.1. *Relations between the shape curvature $\widetilde{\kappa}_1$ and the shape torsion $\widetilde{\kappa}_2$ of the cylindrical helix $\boldsymbol{\gamma}$ generated by a unit speed C^3-plane curve $\boldsymbol{\alpha}$ and the shape curvature $\widetilde{K}$ of $\boldsymbol{\alpha}$ are*

$$\widetilde{\kappa}_1 = \sqrt{1+a^2}|\widetilde{K}| \quad \text{and} \quad \widetilde{\kappa}_2 = \varepsilon a.$$

Proof. In [2, p.285] it was shown that the shape curvature of $\boldsymbol{\alpha}$ is $\widetilde{K} = (1/K)'$. From equations (7) and (1) we conclude that the shape curvature and the shape torsion of $\boldsymbol{\gamma}$ are given by

$$\widetilde{\kappa}_1 = \frac{\frac{d}{ds}(\frac{1}{\kappa_1})}{\|\frac{d\gamma}{ds}\|} = \frac{1}{\sqrt{1+a^2}}\frac{d}{ds}\left(\frac{1+a^2}{|K|}\right) = \sqrt{1+a^2}\left(\frac{1}{|K|}\right)' = \sqrt{1+a^2}|\widetilde{K}| \text{ and } \widetilde{\kappa}_2 = \frac{\kappa_2}{\kappa_1} = \varepsilon a.$$

□

Corollary 2.2.2. *A relation between the focal curvatures of the cylindrical helix $\boldsymbol{\gamma}$ with a parametric equation* (6) *and the shape curvature of the base unit-speed C^3-plane curve $\boldsymbol{\alpha}$ is*

$$\frac{c_2}{c_1} = \frac{\sqrt{1+a^2}}{a}\widetilde{K}. \tag{9}$$

Proof. From (8) the ratio of the focal curvatures is

$$\frac{c_2}{c_1} = -\frac{\sqrt{1+a^2}}{a}\frac{K'}{K^2} = \frac{\sqrt{1+a^2}}{a}\left(\frac{1}{K}\right)' = \frac{\sqrt{1+a^2}}{a}\widetilde{K}.$$

□

3 Focal curves of cylindrical helices generated by unit speed plane curves

Theorem 3.1. *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s), s \in I$ be a unit-speed C^3-plane curve in $\mathbb{E}^2$ and let $K \neq 0$ be the signed curvature of $\boldsymbol{\alpha}$. Suppose that $\boldsymbol{\gamma}(s)$ is the cylindrical helix defined by* (6)*. Then the focal curve $\mathbf{C}_\gamma$ of $\boldsymbol{\gamma}$ possesses a parametrization*

$$\mathbf{C}_\gamma(s) = \boldsymbol{\alpha}(s) + \frac{(1+a^2)K'}{K^3}\mathbf{T} + \frac{1+a^2}{K}\mathbf{N} + \frac{(as+b)aK^3-(1+a^2)K'}{aK^3}\vec{\mathbf{e}}_3, \tag{10}$$

where $\mathbf{T}$ and $\mathbf{N}$ form the Frenet frame of $\boldsymbol{\alpha}$.

Proof. First, we find the Frenet frame $\mathbf{t}$, $\mathbf{n_1}$, $\mathbf{n_2}$ of $\boldsymbol{\gamma}$. From equation (6) and the structure equations of $\boldsymbol{\alpha}$ we obtain the cross vector products

$$\frac{d\boldsymbol{\gamma}}{ds} \times \frac{d^2\boldsymbol{\gamma}}{ds^2} = (\boldsymbol{\alpha}' + a.\vec{\mathbf{e}}_3) \times \boldsymbol{\alpha}'' = (\boldsymbol{\alpha}' + a.\vec{\mathbf{e}}_3) \times K.J\boldsymbol{\alpha}' = K(\boldsymbol{\alpha}' \times J\boldsymbol{\alpha}' + a.\vec{\mathbf{e}}_3 \times J\boldsymbol{\alpha}') = K(\vec{\mathbf{e}}_3 - a.\boldsymbol{\alpha}')$$

$$\left(\frac{d\boldsymbol{\gamma}}{ds} \times \frac{d^2\boldsymbol{\gamma}}{ds^2}\right) \times \frac{d\boldsymbol{\gamma}}{ds} = K(\vec{\mathbf{e}}_3 - a.\boldsymbol{\alpha}') \times (\boldsymbol{\alpha}' + a.\vec{\mathbf{e}}_3) = K(1+a^2)\vec{\mathbf{e}}_3 \times \boldsymbol{\alpha}' = K(1+a^2)J\boldsymbol{\alpha}'.$$

Their norms are $\left\|\frac{d\boldsymbol{\gamma}}{ds} \times \frac{d^2\boldsymbol{\gamma}}{ds^2}\right\| = |K|\sqrt{1+a^2}$ and $\left\|\frac{d\boldsymbol{\gamma}}{ds} \times \frac{d^2\boldsymbol{\gamma}}{ds^2}\right\| . \left\|\frac{d\boldsymbol{\gamma}}{ds}\right\| = (1+a^2)|K|$. From the Frenet formulas with respect to an arbitrary parameter we get the unit principal normal vector

$$\mathbf{n_1} = \frac{\left(\frac{d\boldsymbol{\gamma}}{ds} \times \frac{d^2\boldsymbol{\gamma}}{ds^2}\right) \times \frac{d\boldsymbol{\gamma}}{ds}}{\left\|\frac{d\boldsymbol{\gamma}}{ds} \times \frac{d^2\boldsymbol{\gamma}}{ds^2}\right\| . \left\|\frac{d\boldsymbol{\gamma}}{ds}\right\|} = \frac{K(1+a^2)J\boldsymbol{\alpha}'}{(1+a^2)|K|} = \varepsilon J\boldsymbol{\alpha}', \tag{11}$$

and the unit binormal vector

$$\mathbf{n_2} = \frac{\frac{d\boldsymbol{\gamma}}{ds} \times \frac{d^2\boldsymbol{\gamma}}{ds^2}}{\left\|\frac{d\boldsymbol{\gamma}}{ds} \times \frac{d^2\boldsymbol{\gamma}}{ds^2}\right\|} = \frac{K(\vec{\mathbf{e}}_3 - a.\boldsymbol{\alpha}')}{|K|\sqrt{1+a^2}} = \frac{\varepsilon(\vec{\mathbf{e}}_3 - a.\boldsymbol{\alpha}')}{\sqrt{1+a^2}} \tag{12}$$

Having in mind $\mathbf{T} = \boldsymbol{\alpha}'$ and $\mathbf{N} = J\boldsymbol{\alpha}'$ and using (3), (8), (11) and (12) we obtain (10). □

Definition 3.1. *The focal curve $\mathbf{C}_\gamma$ of $\boldsymbol{\gamma}$ with parametrization* (10) *is called* ***an associated space curve of*** *a unit-speed plane curve $\boldsymbol{\alpha}$ with a nonzero signed curvature.*

The curvature and the torsion of the associated space curve can be expressed in terms of the signed curvature and the shape curvature of $\boldsymbol{\alpha}$.

Theorem 3.2. *The curvature K_1 and the torsion K_2 of the associated curve $\mathbf{C}_\gamma$ of the unit speed C^3-plane curve $\boldsymbol{\alpha}$ with a signed curvature $K \neq 0$ and a shape curvature $\widetilde{K}$ are given by*

$$K_1 = \frac{a^2|K|}{(1+a^2)\left|a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'\right|}, K_2 = \frac{aK}{(1+a^2)\left(a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'\right)}. \tag{13}$$

Proof. Equation (10) can be written in the form

$$\mathbf{C}_\gamma(s) = \boldsymbol{\alpha}(s) - \frac{(1+a^2)\widetilde{K}}{K}\mathbf{T} + \frac{1+a^2}{K}\mathbf{N} + \left((as+b) + \frac{(1+a^2)\widetilde{K}}{aK}\right)\vec{\mathbf{e}}_3. \tag{14}$$

Then, its derivatives with respect to the arc-length parameter s of $\boldsymbol{\alpha}$ are

$$\frac{d\mathbf{C}_\gamma}{ds} = \left(a + \frac{1+a^2}{a}\left(\frac{\widetilde{K}}{K}\right)'\right)(\vec{\mathbf{e}}_3 - a\mathbf{T})$$

$$\frac{d^2\mathbf{C}_\gamma}{ds^2} = \left(a + \frac{1+a^2}{a}\left(\frac{\widetilde{K}}{K}\right)'\right)(-aK.\mathbf{N}) + \frac{1+a^2}{a}\left(\frac{\widetilde{K}}{K}\right)''(\vec{\mathbf{e}}_3 - a\mathbf{T})$$

$$\frac{d^3\mathbf{C}_\gamma}{ds^3} = \left(a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'\right)(K^2.\mathbf{T} - K'\mathbf{N}) - 2(1+a^2)\left(\frac{\widetilde{K}}{K}\right)''K.\mathbf{N} + \frac{1+a^2}{a}\left(\frac{\widetilde{K}}{K}\right)'''(\vec{\mathbf{e}}_3 - a\mathbf{T}).$$

The cross vector product and the triple scalar product are

$$\begin{aligned} &\frac{d\mathbf{C}_\gamma}{ds} \times \frac{d^2\mathbf{C}_\gamma}{ds^2} = \frac{K}{a}\left(a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'\right)^2(\mathbf{T} + a\vec{\mathbf{e}}_3), \\ &\frac{d\mathbf{C}_\gamma}{ds}\frac{d^2\mathbf{C}_\gamma}{ds^2}\frac{d^3\mathbf{C}_\gamma}{ds^3} = < \frac{d\mathbf{C}_\gamma}{ds} \times \frac{d^2\mathbf{C}_\gamma}{ds^2}, \frac{d^3\mathbf{C}_\gamma}{ds^3} > = \frac{K^3}{a}\left(a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'\right)^3 \end{aligned} \tag{15}$$

and the norms of the first derivative and the cross vector product are

$$\left\|\frac{dC_\gamma}{ds}\right\| = \frac{\sqrt{1+a^2}}{a}\left|a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'\right|, \left\|\frac{d\mathbf{C}_\gamma}{ds} \times \frac{d^2\mathbf{C}_\gamma}{ds^2}\right\| = \frac{|K|}{a}\sqrt{1+a^2}\left(a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'\right)^2. \tag{16}$$

From $K_1 = \dfrac{\left\|\frac{d\mathbf{C}_\gamma}{ds} \times \frac{d^2\mathbf{C}_\gamma}{ds^2}\right\|}{\left\|\frac{d\mathbf{C}_\gamma}{ds}\right\|^3}$, $K_2 = \dfrac{\frac{d\mathbf{C}_\gamma}{ds}\frac{d^2\mathbf{C}_\gamma}{ds^2}\frac{d^3\mathbf{C}_\gamma}{ds^3}}{\left\|\frac{d\mathbf{C}_\gamma}{ds} \times \frac{d^2\mathbf{C}_\gamma}{ds^2}\right\|^2}$ and equations (15), (16) we get (13). □

Corollary 3.2.1. *The associated curve $\mathbf{C}_\gamma$ of the unit speed C^3-plane curve $\boldsymbol{\alpha}$ is a cylindrical helix.*

Proof. From (13) we have

$$\frac{K_2}{K_1} = \frac{\varepsilon\delta}{a} = const,$$

where $\varepsilon = sign(K), \delta = sign\left(a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'\right)$. Hence the associated curve is a cylindrical helix. □

Theorem 3.3. *The shape curvature $\widetilde{K}_1$ and the shape torsion $\widetilde{K}_2$ of the associated curve $\mathbf{C}_\gamma$ of the unit speed C^3-plane curve $\boldsymbol{\alpha}$ with a signed curvature $K \neq 0$ and a shape curvature $\widetilde{K}$ are given by*

$$\widetilde{K}_1 = \frac{\sqrt{1+a^2}}{a}\left|\widetilde{K} + \frac{\frac{(1+a^2)}{K}\left(\frac{\widetilde{K}}{K}\right)''}{a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'}\right| \quad \text{and} \quad \widetilde{K}_2 = \frac{\delta}{\varepsilon a}, \tag{17}$$

where $\varepsilon = sign(K)$ and $\delta = sign\left(a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'\right)$.

Proof. From (13) we see that the first focal curvature of $\mathbf{C}_\gamma$ is equal to

$$C_1 = \frac{1}{K_1} = \frac{1+a^2}{a^2|K|}\left|a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'\right|, \tag{18}$$

where the shape curvature $\widetilde{K} = (1/K)'$ of $\boldsymbol{\alpha}$ and $(\widetilde{K}/K)' = (3K'^2 - KK'')/K^4$ are smooth functions depending on the arc-length parameter s of α. After differentiation of (18) with respect to s and some calculations we get

$$\frac{dC_1(s)}{ds} = \frac{1+a^2}{a^2}\left|\widetilde{K}\left((1+a^2)\left(\frac{\widetilde{K}}{K}\right)' + a^2\right) + \frac{1+a^2}{K}\left(\frac{\widetilde{K}}{K}\right)''\right| \tag{19}$$

From $\widetilde{K}_1 = \dfrac{dC_1(s)}{ds}\Big/\left\|\dfrac{dC_\gamma}{ds}\right\|$, $\widetilde{K}_2 = K_2/K_1$ and (19), (16) we obtain (17). □

Definition 3.2. *The orthogonal projection of the associated curve $\mathbf{C}_\gamma$ on the plane $\mathbb{E}^2 = Oxy$ is called **a generalized focal curve** of the plane curve $\boldsymbol{\alpha}$ and denoted by $\boldsymbol{\beta}$.*

From equation (14) we deduce that the generalized focal curve of $\boldsymbol{\alpha}$ is determined by

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) - \frac{(1+a^2)\widetilde{K}}{K}\boldsymbol{\alpha}' + \frac{1+a^2}{K}J\boldsymbol{\alpha}'. \tag{20}$$

4 Applications

4.1 Generalized focal curve of cycloid

Let we consider the cycloid $\boldsymbol{\alpha}_0(t) = (c.(t+\sin t), c.(1+\cos t), 0)$, $\quad c = const > 0, t \in \mathbb{R}$. The arc-length parametrization of one hump of the cycloid $\boldsymbol{\alpha}_0$ is

$$\boldsymbol{\alpha}(s) = \left(\frac{s\sqrt{16c^2 - s^2}}{8c} + 2c\arcsin\left(\frac{s}{4c}\right), \frac{16c^2 - s^2}{8c}, 0\right), -4.c \le s \le 4.c \tag{21}$$

The natural equation of the above cycloid or the so-called Cesàro equation of that plane curve is $R^2 + s^2 = 16c^2$, where $R = 1/|K|$. Then, from (6) the corresponding cylindrical helix has a constant speed parametrization

$$\boldsymbol{\gamma}(s) = \left(\frac{s\sqrt{16c^2 - s^2}}{8c} + 2c\arcsin\left(\frac{s}{4c}\right), \frac{16c^2 - s^2}{8c}, as + b\right) \tag{22}$$

and from (10) the associated curve of $\boldsymbol{\alpha}(s)$ has constant speed parametrization

$$\mathbf{C}_\gamma(s) = \left(\frac{s\sqrt{16c^2 - s^2}}{8c} + 2c\arcsin\left(\frac{s}{4c}\right), \frac{16c^2 - s^2}{8c} - 4c(1+a^2), -\frac{s}{a} + b\right). \tag{23}$$

Finally, the generalized focal curve $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}(s)$ is also a cycloid with an arc-length parametrization

$$\boldsymbol{\beta}(s) = \left(\frac{s\sqrt{16c^2 - s^2}}{8c} + 2c\arcsin\left(\frac{s}{4c}\right), \frac{16c^2 - s^2}{8c} - 4c(1+a^2), 0\right) \tag{24}$$

It is obvious that the curve $\boldsymbol{\beta}$ can be obtained from the curve $\boldsymbol{\alpha}$ via translation determined by the vector $\vec{\mathbf{p}} = (0, -4c(1+a^2), 0)$.

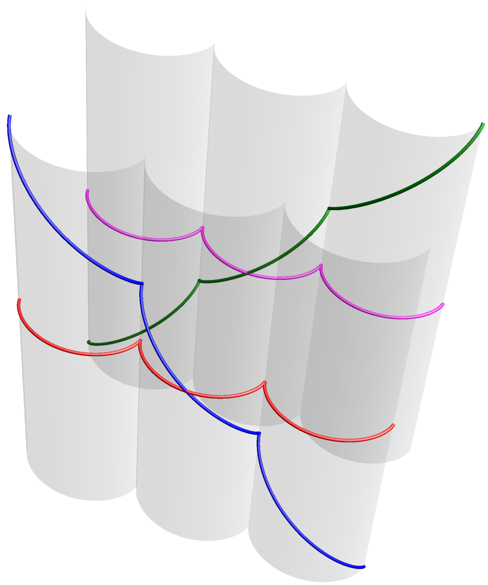

Fig. 1: Cycloid $\boldsymbol{\alpha}$ in red, generalized focal curve $\boldsymbol{\beta}$ in purple, cylindrical helix $\boldsymbol{\gamma}$ in blue, associated curve $\mathbf{C}_\gamma$ of $\boldsymbol{\alpha}$ in green

4.2 Generalized focal curve of closed plane curves with periodic signed curvature

Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a unit speed plane curve with a periodic curvature $K(s) = \frac{m}{n} - \sin s$, where $m \in \mathbb{Z}, n > 1, n \in \mathbb{N}$ and $(m,n) = 1$. Such a curve is introduced and studied in [1]. Then by (2) the function $\theta(s)$ becomes $\theta(s) = \int K(s)ds + \theta_0 = \frac{\pi}{2} + \cos s + \frac{m}{n}s$ and a unit speed parametrisation of $\boldsymbol{\alpha}$ is

$$\boldsymbol{\alpha}(s) = \left(\int \cos\left(\frac{\pi}{2} + \cos s + \frac{m}{n}s\right)ds, \int \sin\left(\frac{\pi}{2} + \cos s + \frac{m}{n}s\right)ds, 0\right).$$

From (6) it follows that the corresponding cylindrical helix has a constant speed parametrization

$$\boldsymbol{\gamma}(s) = \left(\int \cos\left(\frac{\pi}{2} + \cos s + \frac{m}{n}s\right)ds, \int \sin\left(\frac{\pi}{2} + \cos s + \frac{m}{n}s\right)ds, as + b\right)$$

and from (10) the associated curve of $\boldsymbol{\alpha}(s)$ has a parametrization

$$\mathbf{C}_\gamma(s) = \left(-\int \sin\left(\frac{ms}{n} + \cos s\right)ds + A(s), \int \cos\left(\frac{ms}{n} + \cos s\right)ds - B(s), as + b + C(s)\right),$$

where $A(s) = -\frac{(1+a^2)\,n\left(\sin\theta(s)(m - n\sin s)^2 + n^2\cos s.\cos\theta(s)\right)}{(m - n\sin s)^3}$,

$B(s) = \frac{(1+a^2)\,n\left(n^2\cos s.\sin\theta(s) - (m - n\sin s)^2\cos\theta(s)\right)}{(m - n\sin s)^3}$ and $C(s) = \frac{(1+a^2)\,n^3\cos s}{a(m - n\sin s)^3}$.

Then the generalized focal curve $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}(s)$ has a parametrization

$$\boldsymbol{\beta}(s) = \left(-\int \sin\left(\frac{ms}{n} + \cos s\right)ds + A(s), \int \cos\left(\frac{ms}{n} + \cos s\right)ds - B(s), 0\right).$$

It is obvious that associated focal $\mathbf{C}_\gamma$ can be obtained from $\boldsymbol{\gamma}$ by matrix function $T_{\vec{\mathbf{r}}(s)}$ in homogeneous coordinates $(T_{\vec{\mathbf{r}}(s)}) = \begin{pmatrix} 1 & 0 & 0 & A(s) \\ 0 & 1 & 0 & B(s) \\ 0 & 0 & 1 & C(s) \\ 0 & 0 & 0 & 1 \end{pmatrix}$

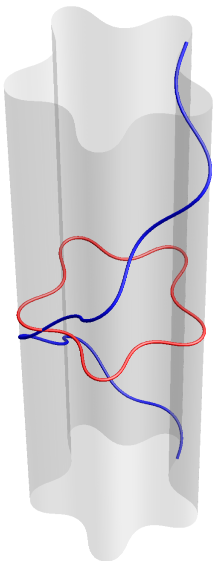

Fig. 2: Closed plane curve $\boldsymbol{\alpha}$ for $m = 1$ and $n = 5$ in red and its cylindrical helix $\boldsymbol{\gamma}$ in blue

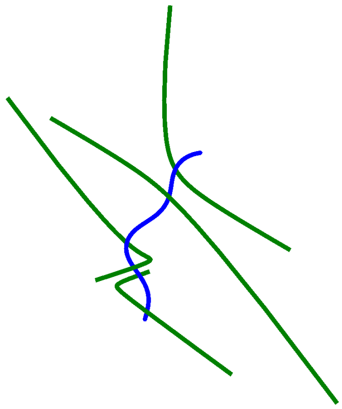

Fig. 3: Focal curve of $\boldsymbol{\gamma}$ in green

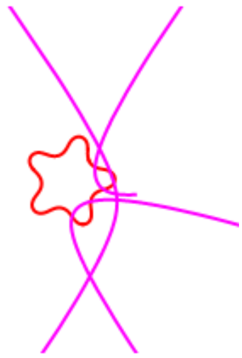

Fig. 4: Plane curve $\boldsymbol{\alpha}$ with a part of its generalized focal curve $\boldsymbol{\beta}$ in purple

REFERENCES:

[1] Arroyo, J., Garay, O.J., Menca, J.J., When is a peroidic function the curvature of a closed plane curve?, Am. Math. Mon. **115** (5), (2008), 405–414.

[2] Encheva, R.P., Georgiev, G.H., Curves on the shape spere, Result. Math. **44** (2003), 279–288.

[3] Encheva, R.P., Georgiev, G.H., Shapes of space curves, J. Geom. Graph. **7** (2003), 145–155.

[4] Encheva, R.P., Georgiev, G.H., Similar Frenet Curves, Result. Math., **55** (3-4) (2009), 359–372. http://www.springerlink.com/content/x6145691n7437r17/

[5] Georgiev, G.H., Encheva, R.P., Dinkova, Cv.L., Geometry of cylindrical curves over plane curves, Applied Mathematical Sciences,**113** (9) (2015), 5637–5649. http://dx.doi.org/10.12988/ams.2015.56456

[6] Gray, A., Abbena, E., Salamon, S., Modern Differential Geometry of Curves and Surfaces, Chapman Hall/CRC, (2006).

[7] Izumiya, S., Takeuchi, N., Generic properties of Helices and Bertrand curves, Journal of Geometry, **74** (2002), 97-109.

[8] Izumiya, S., Takeuchi, N., Special curves and ruled surfaces, Cotributions to Algebra and Geometry, **44** (2003), 203–212.

Georgi Georgiev, Radostina Encheva, Cvetelina Dinkova
Konstantin Preslavsky University of Shumen
E-mails: georgiev@shu.bg, r.encheva@shu.bg, c.dinkova@shu.bg