GENERALIZED FOCAL CURVES OF REGULAR C^3-PLANE CURVES

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ABSTRACT: *In this paper we consider a unit-speed C^3-plane curve and its generalized focal curve. Relations between the Euclidean curvatures, shape curvatures and focal curvatures of the corresponding curves are found. We apply these results to nontrivial self-similar plane curves, an involute of a circle and a catenary.*

KEYWORDS: *Focal curves, Curvatures, Plane curves*

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1 Introduction

The differential-geometric invariants of curves with respect to the group of orientation preserving Euclidean motions or the group of direct similarities have an essential role of recovering the curves up to transformation from the considered groups. The Euclidean differential geometry of curves is completely presented in [7]. The fundamental theorem in the classical curve theory is extended to the orientation preserving similarity group in [3]. Based on the above mentioned theorems, recovering a space curve by a pair of real functions up to an orientation-preserving motion or a similarity transformation is possible (see [7], [2]). We will refer to these invariants as curve determining invariants.

The focal curves and the focal curvatures of smooth curve are explored by Uribe-Vargas in [9]. Relations between Euclidean curvatures and focal curvatures as well as between focal curvatures and so called shape curvatures, that determine the curve up to a orientation preserving similarity, are obtained in [9] and [3], respectively.

There are various methods for constructing new plane or space curves using existing well-known plane curves. An algorithm for creating a new plane curve is given in [6] and it is divided into three steps. First, we obtain a cylindrical helix from a given unit speed plane curve. Second, we determine the focal curve of the obtained cylindrical helix. Third, we find the orthogonal projection of the focal curve in the plane of the first curve. This curve we will call a generalized focal curve. In this paper we consider examples of two well-known plane curves as involute of circle and catenary. In addition, we examine nontrivial self-similar plane curves. We obtain their generalized focal curves and find the relations between the determining invariants of the considered classes of pair-curves. As a result, one may create new curves via these relations without using the aforementioned method. Some properties of the obtained curves are explored.

2 Preliminaries

Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{E}^3$ be a regular C^3-space curve in the three dimensional Euclidean space $\mathbb{E}^3$ with nonzero curvature κ_1 and torsion κ_2. The focal curve of $\boldsymbol{\gamma}$ is the curve given by the equation

$$\mathbf{C}_\gamma(t) = \boldsymbol{\gamma}(t) + c_1(t)\mathbf{n_1}(t) + c_2(t)\mathbf{n_2}(t), \tag{1}$$

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where $\mathbf{n_1}$ is a principal unit normal vector field of $\boldsymbol{\gamma}$, $\mathbf{n_2}$ is a binormal unit vector field of $\boldsymbol{\gamma}$. The coefficients $c_1(t)$ and $c_2(t)$ are smooth functions called focal curvatures of $\boldsymbol{\gamma}$, given by

$$c_1(t) = \frac{1}{\kappa_1(t)}, \quad c_2(t) = -\frac{\frac{d}{dt}\kappa_1(t)}{\| \frac{d\boldsymbol{\gamma}(t)}{dt} \| \kappa_1(t)^2\kappa_2(t)} = \frac{\frac{dc_1(t)}{dt}}{\| \frac{d\boldsymbol{\gamma}(t)}{dt} \| \kappa_2(t)},$$

where $\kappa_1(t)$ and $\kappa_2(t)$ are the curvature and the torsion of $\boldsymbol{\gamma}$ (see [7], p. 241). We denote by $\|.\|$ the length of the vector-functions or vectors.

We define a new space curve $\boldsymbol{\gamma} = \boldsymbol{\gamma}(s)$, $s \in I \subseteq \mathbb{R}$ in the Euclidean space $\mathbb{E}^3$ with parametric representation

$$\boldsymbol{\gamma}(s) = (x(s), y(s), as+b) = \boldsymbol{\alpha}(s) + (as+b)\vec{\mathbf{e}}_3,\ a,b = const \tag{2}$$

with respect to a given right-handed Cartesian coordinate system $O\vec{\mathbf{e}}_1\vec{\mathbf{e}}_2\vec{\mathbf{e}}_3$ in $\mathbb{E}^3$. This curve lies on the generalized cylinder $S_{\boldsymbol{\alpha}}$ with a base plane curve $\boldsymbol{\alpha}(s) = (x(s), y(s)) \subset \mathbb{E}^2 \equiv O\vec{\mathbf{e}}_1\vec{\mathbf{e}}_2$ and rulings parallel to z-axis. Observe that the condition $\kappa_2/\kappa_1 = const$, characterizing cylindrical helixes, holds for the curve $\boldsymbol{\gamma}$. More precisely, $\boldsymbol{\gamma}$ is a cylindrical helix with constant curvatures' ratio $\kappa_2/\kappa_1 = \varepsilon a$, where ε is the sign of the sighed curvature K of $\boldsymbol{\alpha}(s)$ (see [5], p. 5640). In what follows we will call ***$\boldsymbol{\gamma}$ a cylindrical helix generated by the plane curve $\boldsymbol{\alpha}$***. Moreover, this cylindrical helix is a geodesic curve on $S_{\boldsymbol{\alpha}}$ that has to have constant speed $\sqrt{1+a^2}$ and constant slope a with respect to the unit vector $\vec{\mathbf{e}}_3$ (see [8]).

From [6] we have that the focal curve $\mathbf{C}_\gamma$ of $\boldsymbol{\gamma}$ possesses a parametrization

$$\mathbf{C}_\gamma(s) = \boldsymbol{\alpha}(s) + \frac{(1+a^2)K'}{K^3}\mathbf{T} + \frac{1+a^2}{K}\mathbf{N} + \frac{(as+b)aK^3 - (1+a^2)K'}{aK^3}\vec{\mathbf{e}}_3, \tag{3}$$

where $\mathbf{T}$ and $\mathbf{N}$ form the Frenet frame of $\boldsymbol{\alpha}$ and the derivative about an arc-length parameter is denoted by "$'$". We will call the curve $\mathbf{C}_\gamma$ **an associated space curve of unit-speed plane curve $\boldsymbol{\alpha}$** with nonzero signed curvature. The relations between the Euclidean curvatures K_1, K_2 and K of the corresponding pair-curves $\mathbf{C}_\gamma$ and $\boldsymbol{\alpha}$, respectively, are given by the equalities

$$K_1 = \frac{a^2|K|}{(1+a^2)\left|a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'\right|}, K_2 = \frac{aK}{(1+a^2)\left(a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'\right)}, \tag{4}$$

where $\widetilde{K} = \left(\dfrac{1}{K}\right)'$ is the shape curvature of $\boldsymbol{\alpha}$. Moreover, the relations between the shape curvatures $\widetilde{K}_1$, $\widetilde{K}_2$ and $\widetilde{K}$ of the corresponding pair-curves $\mathbf{C}_\gamma$ and $\boldsymbol{\alpha}$, respectively, are given by the equalities

$$\widetilde{K}_1 = \frac{\sqrt{1+a^2}}{a}\left|\widetilde{K} + \frac{\frac{(1+a^2)}{K}\left(\frac{\widetilde{K}}{K}\right)''}{a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'}\right| \quad \text{and} \quad \widetilde{K}_2 = \frac{\delta}{\varepsilon a}, \tag{5}$$

where $\varepsilon = sign(K)$, $\delta = sign\left(a^2 + (1+a^2)\left(\frac{\widetilde{K}}{K}\right)'\right)$ and a is the constant slope of $\boldsymbol{\gamma}$.

3 A generalized focal curve of plane curve.

Definition 3.1. *The orthogonal projection of the associated curve $\mathbf{C}_\gamma$, defined by (3), on the plane $\mathbb{E}^2 \equiv O\vec{\mathbf{e}}_1\vec{\mathbf{e}}_2$ is called **a generalized focal curve of the plane curve $\boldsymbol{\alpha}$**.*

We denote that curve by $\boldsymbol{\beta}$. From the equation (3) we deduce that the generalized focal curve of $\boldsymbol{\alpha}$ is determined by

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) - \frac{(1+a^2)\widetilde{K}}{K}\boldsymbol{\alpha}' + \frac{1+a^2}{K}J\boldsymbol{\alpha}', \tag{6}$$

where J is the complex structure of $\mathbb{E}^2$.

Theorem 3.1. *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, $s \in I$ be a unit-speed C^3-plane curve in $\mathbb{E}^2$ with Euclidean and shape curvatures $K_\alpha \neq 0$ and $\widetilde{K}_\alpha$, respectively. Suppose that $\boldsymbol{\beta} = \boldsymbol{\beta}(s)$ is the corresponding generalized focal curve of $\boldsymbol{\alpha}$. Then the Euclidean and shape curvatures $K_\beta \neq 0$ and $\widetilde{K}_\beta$ of $\boldsymbol{\beta}$ are given by*

$$K_\beta = \frac{K_\alpha}{\left|a^2 + (1+a^2)\left(\frac{\widetilde{K}_\alpha}{K_\alpha}\right)'\right|}, \widetilde{K}_\beta = \left(\ln\left|a^2 + (1+a^2)\left(\frac{\widetilde{K}_\alpha}{K_\alpha}\right)'\right|\right)' R_\alpha + \widetilde{K}_\alpha, \tag{7}$$

where $R_\alpha = \dfrac{1}{K_\alpha}$ is the radius of curvature of α and a is the constant slope of $\boldsymbol{\gamma}$.

Proof. From (6) we find the derivative of $\boldsymbol{\beta}(s)$ about an arc-length parameter s and we obtain that

$$\frac{d\boldsymbol{\beta}(s)}{ds} = -\left(a^2 + (1+a^2)\left(\frac{\widetilde{K}_\alpha}{K_\alpha}\right)'\right)\alpha'(s), \left\|\frac{d\boldsymbol{\beta}(s)}{ds}\right\| = \left|a^2 + (1+a^2)\left(\frac{\widetilde{K}_\alpha}{K_\alpha}\right)'\right|.$$

From the Frenet-Seret equations about an arbitrary parameter we get that the unit tangent vector $\boldsymbol{\beta}'(s)$ of $\boldsymbol{\beta}$ is

$$\boldsymbol{\beta}'(s) = \frac{d\boldsymbol{\beta}(s)}{ds} / \left\|\frac{d\boldsymbol{\beta}(s)}{ds}\right\| = -\delta\alpha'(s), \delta = sign\left(a^2 + (1+a^2)\left(\frac{\widetilde{K}_\alpha}{K_\alpha}\right)'\right).$$

If $\boldsymbol{\beta}''$ is the second derivative about the arc-length parameter of the curve $\boldsymbol{\beta}$ we have that

$$\boldsymbol{\beta}''(s) = \frac{d\boldsymbol{\beta}'(s)}{ds} / \left\|\frac{d\boldsymbol{\beta}(s)}{ds}\right\| = -\delta\boldsymbol{\alpha}'' / \left\|\frac{d\boldsymbol{\beta}(s)}{ds}\right\| = K_\alpha(-\delta J\boldsymbol{\alpha}') / \left\|\frac{d\boldsymbol{\beta}(s)}{ds}\right\|$$

and from $-\delta J\boldsymbol{\alpha}' = J\boldsymbol{\beta}'$ and the structure equations

$$\boldsymbol{\beta}'' = K_\beta . J\boldsymbol{\beta}', J\boldsymbol{\beta}'' = -K_\beta . \boldsymbol{\beta}'$$

of $\boldsymbol{\beta}$ we obtain that $K_\beta = \dfrac{K_\alpha}{\left\|\frac{d\boldsymbol{\beta}(s)}{ds}\right\|} = \dfrac{K_\alpha}{\left|a^2 + (1+a^2)\left(\frac{\widetilde{K}_\alpha}{K_\alpha}\right)'\right|}$. Hence, we get

$$\widetilde{K}_\beta = \frac{dK_\beta^{-1}}{ds} / \left\|\frac{d\boldsymbol{\beta}(s)}{ds}\right\| = \frac{d}{ds}\left(\left\|\frac{d\boldsymbol{\beta}(s)}{ds}\right\| K_\alpha^{-1}\right) / \left\|\frac{d\boldsymbol{\beta}(s)}{ds}\right\| = \left(\ln\left\|\frac{d\boldsymbol{\beta}(s)}{ds}\right\|\right)' R_\alpha + \widetilde{K}_\alpha.$$

□

Corollary 3.1.1. *The relations between the Euclidean curvatures K_1 and K_2 of the associated with $\boldsymbol{\alpha}$ space curve $\mathbf{C}_\gamma$ and the signed curvature K_β of the corresponding generalized focal curve $\boldsymbol{\beta}$ are*

$$K_1 = \frac{a^2 \varepsilon K_\beta}{1+a^2} \quad and \quad K_2 = \frac{a\delta K_\beta}{1+a^2}. \tag{8}$$

Proof. The proof follows immediately from the first equation of (7) and (4). □

Theorem 3.2. *Suppose that $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, $s \in I$ is a unit-speed C^3-plane curve with signed curvature $K_\alpha \neq 0$ and $\boldsymbol{\gamma}[a,b](s) = \boldsymbol{\alpha}(s) + (as+b)\vec{\mathbf{e}}_3$ be the corresponding geodesic curve on the generalized cylinder S_α. Let $\boldsymbol{\beta} = \boldsymbol{\beta}(s)$ be the generalized focal curve of $\boldsymbol{\alpha}$ with signed curvature $K_\beta \neq 0$. For the one-parameter family of curves $\boldsymbol{\gamma}[1,b]$ the following conditions are equivalent:*

(i) *$\boldsymbol{\alpha}$ is one of the following curves: a circle, an involute of a circle, or a cycloid;*

(ii) $K_\alpha = K_\beta$.

Proof. If $\boldsymbol{\alpha}$ is a circle with radius r then $K_\alpha = 1/r$ and $\widetilde{K}_\alpha = 0$. From $a = 1$ and the first equation of (7) we obtain that $K_\alpha = K_\beta$. Now we consider the involutes of a circle with natural equation $R_\alpha^2 = 2ps + 2q$, $p, q = const$, where $R_\alpha = \dfrac{1}{K_\alpha}$. After differentiation about an arc-length parameter s we have $2R_\alpha R_\alpha^{'} = 2p$. Hence, $\widetilde{K}_\alpha / K_\alpha = p$ and $(\widetilde{K}_\alpha / K_\alpha)^{'} = 0$. Applying (7) we get $K_\alpha = K_\beta$. Finally, let us consider the equation $R_\alpha^2 = -s^2 + 2ms + 2n$, $m, n = const$ that is the natural equation of cycloids. Then $2R_\alpha R_\alpha^{'} = -2s + 2m \Leftrightarrow R_\alpha R_\alpha^{'} = -s + m \Leftrightarrow \dfrac{1}{K_\alpha}\left(\dfrac{1}{K_\alpha}\right)^{'} = \dfrac{\widetilde{K}_\alpha}{K_\alpha} = -s + m \Leftrightarrow$ $(\widetilde{K}_\alpha / K_\alpha)^{'} = -1$ and from (7) it follows **(ii)**.

Conversely, let $a = 1$ and $K_\alpha = K_\beta \Leftrightarrow \left|1 + 2\left(\dfrac{\widetilde{K}_\alpha}{K_\alpha}\right)^{'}\right| = 1 \Leftrightarrow \left(\dfrac{\widetilde{K}_\alpha}{K_\alpha}\right)^{'} = 0 \ or \ \left(\dfrac{\widetilde{K}_\alpha}{K_\alpha}\right)^{'} = -1$. The curves satisfying the first differential equation have natural equation $R_\alpha^2 = 2ps + 2q$, $p, q = const$ and they are circles for $p = 0, q > 0$ or involutes of circles for $p > 0$. The curves that satisfy the second differential equation are cycloids with natural equation $R_\alpha^2 = -s^2 + 2ms + 2n$, $m, n = const$. □

4 Invariants of self-similar cylindrical curves over plane curves

A Frenet space curve $\boldsymbol{\gamma} : I \to \mathbb{E}^3$ is called *a self-similar curve*, if $\boldsymbol{\gamma}$ is an orbit under the action of one-parameter group of direct similarities. Hence, a Frenet curve in $\mathbb{E}^3$ is self-similar if and only if its shape invariants $\widetilde{\kappa}_1$, $\widetilde{\kappa}_2$ are constants. The unique *self-similar* curves in the Euclidean plane are the straight lines, the circles and the logarithmic spirals. In addition, the conic helixes complete the set of self-similar curves in the three-dimensional Euclidean space. Straight lines are excluded from nontrivial self-similar curves.

Lemma 4.1. *The geodesic curve $\boldsymbol{\gamma}$ on a generalized cylinder over a nontrivial self-similar unit-speed C^3-plane curve $\boldsymbol{\alpha}$ is also a self-similar curve.*

Proof. Let $\boldsymbol{\alpha}$ be a nontrivial self-similar unit-speed C^3-plane curve. Then the shape curvature $\widetilde{K}$ of $\boldsymbol{\alpha}$ is a constant and from Corollary 2.2.1 in [6] we have that the shape curvatures $\widetilde{\kappa}_1 = \sqrt{1+a^2}|\widetilde{K}|$ and $\widetilde{\kappa}_2 = \varepsilon a$ are also constants. Hence, $\boldsymbol{\gamma}$ is a self-similar curve. □

Lemma 4.2. *The relation between shape curvature of associated space curve* $\mathbf{C}_\gamma$ *and shape curvature of nontrivial self-similar unit-speed* C^3*-plane curve* $\boldsymbol{\alpha}$ *is*

$$\widetilde{K}_1 = \frac{\sqrt{1+a^2}}{a}|\widetilde{K}|.$$

Proof. It follows immediately from (5) and $\widetilde{K} = const$. □

Theorem 4.3. *The associated space curve* $\mathbf{C}_\gamma$ *of a nontrivial self-similar unit-speed* C^3*-plane curve* α *is also a self-similar curve.*

Proof. Let $\boldsymbol{\alpha}$ be a self-similar curve. Then the shape curvature $\widetilde{K}$ of $\boldsymbol{\alpha}$ is a constant. From Lemma 4.2 and the second equation of (5) it follows that $\mathbf{C}_\gamma$ is also a self-similar curve. □

Corollary 4.3.1. *The relations between the shape curvatures* $\widetilde{\kappa}_1, \widetilde{\kappa}_2$ *of a self-similar cylindrical curve* $\boldsymbol{\gamma}$ *and the shape curvatures* $\widetilde{K}_1, \widetilde{K}_2$ *of the associated space curve* $\mathbf{C}_\gamma$ *are given by*

$$\widetilde{K}_1 = \frac{\widetilde{\kappa}_1}{|\widetilde{\kappa}_2|} \quad \textit{and} \quad \widetilde{K}_2 = \frac{1}{\widetilde{\kappa}_2}. \tag{9}$$

Proof. Since $\widetilde{K} = const$ then $\left(\dfrac{\widetilde{K}}{K}\right)' = \widetilde{K}^2 > 0$ and $\delta = sign\Big(a^2 + (1+a^2)\Big(\frac{\widetilde{K}}{K}\Big)'\Big)$ is positive. Then applying the relations $\widetilde{\kappa}_1 = \sqrt{1+a^2}|\widetilde{K}|$ and $\widetilde{\kappa}_2 = \varepsilon a$ from Corollary 2.2.1 in [6], Lemma 4.2 and $\widetilde{K}_2 = \dfrac{\delta}{\varepsilon a}$ we obtain (9). □

5 Examples of geodesic cylindrical curves and their focal curves.

5.1 Involute of a circle

We consider the involute of a circle $k(O,r)$ with unit-speed parametrization

$$\boldsymbol{\alpha}(s) = \left(r\cos\sqrt{\frac{2s}{r}} + \sqrt{2rs}\,\sin\sqrt{\frac{2s}{r}}, r\sin\sqrt{\frac{2s}{r}} - \sqrt{2rs}\,\cos\sqrt{\frac{2s}{r}}\right). \tag{10}$$

Let $\boldsymbol{\gamma}$ be the corresponding geodesic with parametric equation

$$\boldsymbol{\gamma}[a,b](s) = \boldsymbol{\alpha}(s) + (as+b)\vec{e}_3. \tag{11}$$

Using (3) we obtain that the focal curve C_γ associated with $\boldsymbol{\alpha}$ has parametric equation

$$C_\gamma(s) = -a^2\,\boldsymbol{\alpha}(s) + (as+d)\vec{e}_3, \text{ where } d = \frac{r(1+a^2)}{a} + b = const. \tag{12}$$

The orthogonal projection of C_γ onto coordinate plane Oxy is the curve $\boldsymbol{\beta}(s) = -a^2\boldsymbol{\alpha}(s)$. It has constant speed and it is obvious that the focal curve C_γ is a geodesic over the generalized cylinder $S_\beta(u,v) = \boldsymbol{\beta}(u) + (av+d)\vec{e}_3$ with base plane curve $\boldsymbol{\beta}$.
Let us introduce homogeneous coordinates (x,y,z,t) in $\mathbb{E}^3 \equiv O\vec{\mathbf{e}}_1\vec{\mathbf{e}}_2\vec{\mathbf{e}}_3$ by $X = \dfrac{x}{t}, Y = \dfrac{y}{t}, Z = \dfrac{z}{t}, t \neq 0$, where (X,Y,Z) are Cartesian coordinates in $\mathbb{E}^3$ and $\Omega : t = 0$ is the plane at infinity.

Theorem 5.1. *Let* $\boldsymbol{\alpha}$ *be a* C^3*-plane curve with parametrization* (10) *and* $\boldsymbol{\gamma}[a,b]$ *is the corresponding geodesic with parametrization* (11)*. Let the focal curve* C_γ *associated with* $\boldsymbol{\alpha}$ *has parametric equation* (12)*. Then* $\Phi : \boldsymbol{\gamma}[a,b](s) \to C_\gamma(s)$ *is an affine transformation with double point at infinity* $R(0,0,1,0)$ *and line of double points at infinity* $u_\infty : \begin{cases} z=0 \\ t=0 \end{cases}$ *in homogeneous coordinates. For the one-parameter family of curves* $\boldsymbol{\gamma}[1,b]$ *the transformation* Φ *is a direct similarity.*

Proof. The matrix representation of Φ in homogeneous coordinates is

$$(\lambda' x', \lambda' y', \lambda' z', \lambda' t') = (x,y,z,t).\begin{pmatrix} -a^2 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & r(1+a^2)/a & 1 \end{pmatrix}, \tag{13}$$

where the matrix A_Φ of Φ has nonzero determinant, i.e. $\det A_\Phi = a^4 \neq 0$. It is clear that Φ is a linear transformation. The double points of Φ are the nonzero solution of the homogeneous system

$$\left| \begin{array}{llllr} (-\lambda - a^2)x & & & & =0 \\ & (-\lambda - a^2)y & & & =0 \\ & & (-\lambda+1)z+ & (r(1+a^2)/a)t & =0 \\ & & & (-\lambda+1)t & =0 \end{array} \right. . \tag{14}$$

The last system has nonzero solution if and only if its determinant satisfies the equality

$$\begin{vmatrix} -\lambda - a^2 & 0 & 0 & 0 \\ 0 & -\lambda - a^2 & 0 & 0 \\ 0 & 0 & -\lambda+1 & r(1+a^2)/a \\ 0 & 0 & 0 & -\lambda+1 \end{vmatrix} = (\lambda+a^2)^2(1-\lambda)^2 = 0. \tag{15}$$

For $\lambda = 1$ we get that the double points of Φ satisfy the system $\left| \begin{array}{cl} (1+a^2)x & =0 \\ (1+a^2)y & =0 \\ (r(1+a^2)/a)t & =0 \\ 0.z & =0 \end{array} \right.$, i.e. the point $R(0,0,1,0)$ is a double point of Φ. Let δ be a plane with homogeneous coordinates $\delta[A,B,C,D]$. To find the double planes of Φ we solve the system $\left| \begin{array}{cl} (1+a^2)A & =0 \\ (1+a^2)B & =0 \\ (r(1+a^2)/a)C & =0 \\ 0.D & =0 \end{array} \right.$. Therefore $A=B=C=0, D\neq 0$ and the plane at infinity $\Omega : t=0$ is a double plane to Φ, i. e. Φ is an affine transformation. For $\lambda = -a^2$ we get the system $\left| \begin{array}{cl} 0.x & =0 \\ 0.y & =0 \\ (1+a^2).z + (r(1+a^2)/a)t & =0 \\ (1+a^2).t & =0 \end{array} \right.$, i.e. all points of the line at infinity $u_\infty : \begin{cases} z=0 \\ t=0 \end{cases}$ of Oxy plane are double points of Φ. In this case the double planes satisfy the system $\left| \begin{array}{cl} 0.A & =0 \\ 0.B & =0 \\ (1+a^2).D + (r(1+a^2)/a).C & =0 \\ (1+a^2).C & =0 \end{array} \right.$ and they have equation

$\delta : Ax + By = 0, A^2 + B^2 \neq 0$. Now let us find a condition when the affine transformation Φ is a direct similarity. According to [1], if the shape curvatures of the corresponding curves are equal, the curves are similar and have the same orientation. After some calculations and simplifications we get that $K = 1/\sqrt{2rs}$ and $\widetilde{K} = r/\sqrt{2rs}$, where K and $\widetilde{K}$ are the signed curvature and the shape curvature of the base plane curve $\boldsymbol{\alpha}$. Since $\widetilde{K}/K = r$ is a constant then $(\widetilde{K}/K)' = (\widetilde{K}/K)'' = 0$. Applying the last equality in (5) we have that the shape curvatures of the associated focal curve C_γ are $\widetilde{K}_1 = \dfrac{\sqrt{1+a^2}}{a}|\widetilde{K}| = \dfrac{r\sqrt{1+a^2}}{a\sqrt{2rs}}$ and $\widetilde{K}_2 = \dfrac{1}{a}$. From Corollary 2.2.1 in [6] the shape curvatures of the geodesic cylindrical curve $\boldsymbol{\gamma}$ are $\widetilde{\kappa}_1 = \sqrt{1+a^2}|\widetilde{K}| = \dfrac{r\sqrt{1+a^2}}{\sqrt{2rs}}$ and $\widetilde{\kappa}_2 = \varepsilon a$. Since $\varepsilon = sign K = +1$ it is clear that $\widetilde{K}_1 = \widetilde{\kappa}_1$ and $\widetilde{K}_2 = \widetilde{\kappa}_2$ if and only if the constant slope a of $\boldsymbol{\gamma}$ is equal to one. $\square$

Corollary 5.1.1. *Let $\boldsymbol{\alpha}$ be an involute of a circle, defined by equations (10), and $\boldsymbol{\beta}$ is the generalized focal curve of $\boldsymbol{\alpha}$. The transformation $\phi : \boldsymbol{\alpha}(s) \to \boldsymbol{\beta}(s)$ is a homothety with coefficient $-a^2$ and the curve $\boldsymbol{\beta}(s)$ is an involute of a circle $\bar{k}(O, \bar{r})$, where $\bar{r} = ra^2$. If Φ is a direct similarity then ϕ is a central symmetry with center the origin.*

Proof. From $\boldsymbol{\beta}(s) = -a^2\boldsymbol{\alpha}(s)$ it is obvious that ϕ is a homothety with coefficient $-a^2$ and the curve $\boldsymbol{\beta}(s)$ is an involute of a circle $\bar{k}(O, \bar{r})$, where $\bar{r} = ra^2$. Next, the proof follows immediately from Theorem 5.1. $\square$

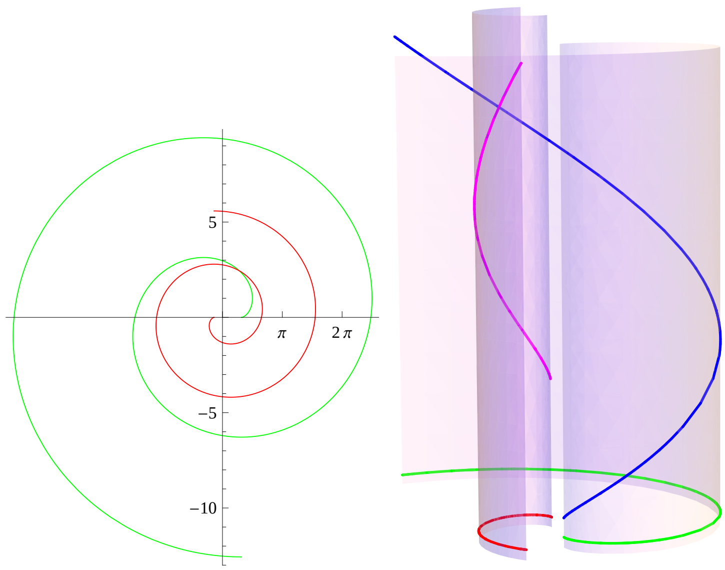

Fig. 1: An involute of circle in green, curve β in red, space curve γ in blue, its focal curve C_γ in purple

Another interesting maps between space curve $\boldsymbol{\gamma}$ and its corresponding focal curve C_γ are explored in [4].

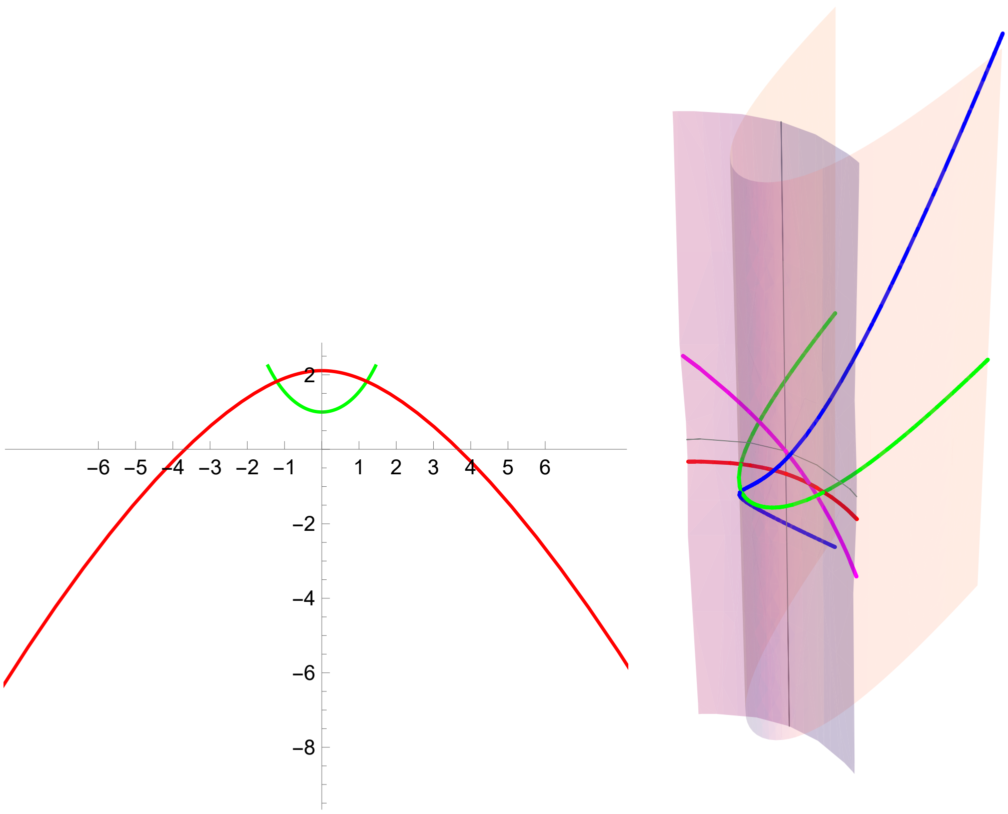

Fig. 2: The catenary $\boldsymbol{\alpha}$ in green, its generalized focal curve β in red, geodesic space curve γ in blue, associated space curve C_γ in purple

5.2 Catenary

Now let us consider the catenary with unit-speed parametrization

$$\boldsymbol{\alpha}(s) = \left(c.\ln\frac{s+\sqrt{c^2+s^2}}{c}, \sqrt{c^2+s^2}\right),$$

where $c = const > 0$ and let $\boldsymbol{\gamma}(s) = \boldsymbol{\alpha}(s) + (as+b)\vec{e}_3$ be the corresponding geodesic curve. Using (3) we obtain that the focal curve associated with $\boldsymbol{\alpha}$ has parametric equations

$$C_\gamma(s) : \begin{cases} x(s) = \dfrac{-3\left(1+a^2\right)s\sqrt{c^2+s^2}}{c} + c\ln\dfrac{s+\sqrt{c^2+s^2}}{c} \\ y(s) = \dfrac{\sqrt{c^2+s^2}\left(\left(2+a^2\right)c^2 - 2\left(1+a^2\right)s^2\right)}{c^2} \\ z(s) = b + as + \dfrac{2\left(1+a^2\right)s\left(c^2+s^2\right)}{ac^2} \end{cases}.$$

Therefore, the generalized focal curve $\boldsymbol{\beta}(s)$ of $\boldsymbol{\alpha}(s)$ has parametric equation

$$\boldsymbol{\beta}(s) = \left(c\ln\frac{s+\sqrt{c^2+s^2}}{c} - \frac{3\left(1+a^2\right)s\sqrt{c^2+s^2}}{c}, \frac{\sqrt{c^2+s^2}\left(\left(2+a^2\right)c^2 - 2\left(1+a^2\right)s^2\right)}{c^2}\right)$$

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