ON TWO MODIFICATIONS OF WEIERSTRASS-DOCHEV'S ITERATIVE METHOD FOR SOLVING POLYNOMIAL EQUATIONS

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ABSTRACT: *In this work we investigate two new modifications of the known Weierstrass iterative method for solving polynomial equations. The new iterative methods are obtained by using the companion matrix method and similarity transformation of companion matrices. One dimensional case of the presented methods is also investigated. Analysis of convergence is also provided.*

KEYWORDS: *Polynomial equations, Weierstrass-Dochev's iterative method, Companion matrix method, Simultaneous rootfinding methods.*

1. INTRODUCTION

Let us consider a monic polynomial

$$(1.1) \qquad p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 ,$$

with simple real or complex zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$ and let $x_1, x_2, \ldots, x_n$ be approximations of those zeros. One of the most popular iterative methods for approximating, simultaneously, all the zeros of polynomials is presented by the formula

$$(1.2) \qquad x_i^{k+1} = x_i^k - \frac{p(x_i^k)}{\prod_{j \neq i}^{n} \left(x_i^k - x_j^k\right)}, \quad \text{where} \quad i = 1, \ldots, n \quad \text{and} \quad k = 0,1,2,\ldots .$$

It is known as *Weierstrass' method* (or *WDK method, see [1]*) and the quotient

$$(1.3) \qquad W_i = W_i(x) = \frac{p(x_i)}{\prod_{j \neq i}^{n} \left(x_i - x_j\right)}$$

is called *Weierstrass correction*. Algorithm (1.2) has been rediscovered several times by different authors like Durand [2], Dochev [3], Borsch-Supan [4], Kerner [5], Prešić [6] and by many other authors. Dochev was the first who proved the quadratic convergence of this algorithm.

Our goal in this work is promote and explore two new modifications of the WDK method obtained by using the companion matrix method. The paper is organized as follows. In Section 2 the new modifications of WDK method are presented. Analysis of convergence is provided in Section 3.One dimensional case is explored in Section 4.

2. PRELIMINARY RESULTS

In our previous works (see [7,8]) we investigate the companion matrix method and similarity transformations between some companion matrices. Let us consider a part of the results.

2.1 Companion matrices and some similarity transformations

It is well known that the zeros of a polynomial can be obtained by computing the eigenvalues of the corresponding companion matrix.

Definition 2.1 *An $n \times n$ matrix $A=A_p$ is called generalized companion matrix for a polynomial p(x) presented in (1.1) if the zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$ of p is the set of the eigenvalues of A.*

Thus the problem of finding the zeros of $p(x)$ can be transformed to the problem of finding the eigenvalues of A, i.e eigenvalue problem of A, where matrix methods can be applied. There are many companion matrices for calculating polynomial zeros.

Probably the best known examples of generalized companion matrices whose eigenvalues are precisely the roots of the corresponding polynomial is the Frobenius companion matrix, which is defined directly in terms of the coefficients of the polynomial, is. For the polynomial (1.1) the Frobenius matrix is given by

$$(2.1) \quad F = F_p = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{pmatrix}.$$

It is easy to verify that the characteristic polynomial of F satisfies the equation

$$(2.2) \quad \left| xI - F_p \right| = p(x)$$

and the eigenpair of F_p is (α_i, v^i), where $v^i = \left(\alpha_i^{j-1}\right)_{j=1}^{n}$ is the right eigenvector corresponding to α_i.
The corresponding eigenvector matrix of the Frobenius matrix (2.1) is

$$(2.3) \quad V = V(\alpha_1,..,\alpha_n) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix}.$$

This matrix is known as *Vandermonde matrix* and it is invertible if and only if all the $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct. Then the diagonalization of Frobenius matrix is

$$(2.4) \quad V^{-1}(\alpha) F_p V(\alpha) = \Lambda = diag(\alpha_i),$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.
In [7] (see Theorem 2), we prove the following theorem.
Theorem 2.1 *Let consider a monic polynomial* $\boldsymbol{p}(x)$ *of degree n (1.1), with simple zeros* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *and corresponding companion matrix (2.1). If* $x_1, x_2, \ldots, x_n$ *are distinct approximations of the zeros, then the matrix*

(2.5) $G = V^{-1}(x)F_pV(x) = D - v^Tu^T$, $D = diag\{x_i\}$,

where u *and* v *are defined by*

(2.6) $u = (p(x_1), p(x_2), \ldots, p(x_n))^T$ and $v = \left(\frac{1}{\prod_{j=2}^{n}(x_1 - x_j)}, \ldots, \frac{1}{\prod_{j=1}^{n-1}(x_n - x_j)} \right)$

and $V(x)$ *is the Vandermonde matrix of* $x = (x_1, x_2, \ldots, x_n)$, *is diagonal plus rank one companion matrix of* $\boldsymbol{p}$.

For the entries of the matrix G in Theorem 2.1 we have

$$G_{ii} = x_i - \frac{p(x_i)}{\prod_{j \neq i}^{n}(x_i - x_j)} = x_i - W_i(x) \text{ and } G_{ij} = -\frac{p(x_j)}{\prod_{j \neq i}^{n}(x_i - x_j)} \text{ for } i \neq j .$$

For some recent results on this topic see for example [10], [11], [12], [13] and references therein. Also the following corollary is fulfilled.

Corollary 2.1 *From Theorem 2.1 it follows that the matrix iterative sequence*

(2.7) $G^{k+1} = V^{-1}(x^k)F_pV(x^k)$, $k = 0,1,2,\ldots$

where $x^k = (x_1^k, x_2^k, \ldots, x_n^k)$ *and* $x_i^k = G_{ii}^k$ *for k>0 (i.e.* x_i^k *is the diagonal entry of matrix **G** at row **i**), will converge to the diagonal matrix* $\Lambda = diag(\alpha_i)$.

Therefore, we can consider algorithm (2.7) as a matrix version of the WDK iterative method (1.2).

Using that the equation (2.4) is equivalent to the following equation

(2.8) $V^{-1}(\alpha)F_p^{-1}V(\alpha) = \Lambda^{-1} = diag\left(\frac{1}{\alpha_i}\right)$ where $\alpha_i \neq 0$ for $i = 1, \ldots, n$.

For the diagonal entries of the following matrix

(2.9) $H = V^{-1}(x)F^{-1}V(x) = (H_{ij})$

is obtained the form

(2.10) $H_{ii} = \frac{1}{x_i}\left(1 - \frac{p(x_i)}{a_0}\prod_{j \neq i}^{n}\frac{x_j}{(x_j - x_i)}\right)$, $i = 1, \ldots, n$,

(see Theorem 4.1 in [8]).

By analogy of Corollary 2.1 we consider the matrix iterative sequence

(2.11) $H^{k+1} = V^{-1}(x^k)F_p^{-1}V(x^k)$, $k = 0,1,2,\ldots$

where $x^k = (x_1^k, x_2^k, \ldots, x_n^k)$ and $x_i^k = H_{ii}^k$ for $k > 0$, (i.e. x_i^k is the diagonal entry of matrix H at row i) and we expect that this sequence will converge to the diagonal matrix $\Lambda = diag\left({}^{1}\!/_{\alpha_i} \right)$.

2.2 First modification

In [8] using (2.10)-(2.11) we suggest the following iterative function for simultaneous approximation of all the zeros of polynomials

$$(2.12)\quad x_i^{k+1} = \frac{x_i^k}{1 - \dfrac{p(x_i^k)}{a_0} \prod_{j \neq i}^{n} \dfrac{x_j^k}{\left(x_j^k - x_i^k\right)}},\ i = 1, \ldots, n,\ (\text{where } a_0 \neq 0)$$

called *inverse WDK method*.

2.3 Second modification

Using the Vietas formula for the monic polynomial defined by (1.1)

$$a_0 = (-1)^n \prod_{j=1}^{n} \alpha_j\ ,$$

the iterative function (2.12) is modified as

$$(2.13)\quad x_i^{k+1} = \frac{x_i^k}{1 + \dfrac{W_i^k}{x_i^k}},$$

where $i = 1, \ldots, n$ and $W_i^k = W(x^k)$ is the Weierstrass correction. The iteration (2.13) is called *modified inverse WDK method.*

3. ANALYSIS OF CONVERGENCE

3.1 First modification

Let us consider the roots of polynomial (1.1) as a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in C^n$. Then we say that α is a root-vector of polynomial *p(x)*. In [9], J. Traub presents the following result.

Theorem 3.1 *Let* $\varphi_i = \varphi_i(x_1, \ldots, x_n)$ *and the vector function* $\varphi(x) = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ *has a fixed point at the point* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. *If for an integer r>1*

$$\frac{\partial^s \varphi_i(\alpha)}{\partial z_{s_1} \ldots \partial z_{s_k}} = 0;\ \ 1 \leq s \leq r-1;\ \ 1 \leq i, s_1, \ldots, s_k \leq n$$

and for at least one combination $i, s_1, \ldots, s_k$

$$\frac{\partial^r \varphi_i(\alpha)}{\partial z_{s_1} \ldots \partial z_{s_k}} \neq 0,$$

is fulfilled, then for sufficiently close approximation $x^0 = (x_1^0, \ldots, x_n^0)$ *of* $\alpha = (\alpha_1, \ldots, \alpha_n)$, *the iteration*

$$x^{k+1} = \varphi(x^k),\ \ k = 0,1,2,\ldots$$

converges to α *with* ***r*** *order of convergence.*

Now let us consider the iterative functions (2.12) as the vector function:

(2.14) $\varphi(x) = (\varphi_1(x), \dots, \varphi_n(x))$, where $\varphi_i(x) = \dfrac{x_i}{1 - \dfrac{p(x_i)}{a_0} \prod_{j \neq i}^{n} \dfrac{x_j}{(x_j - x_i)}}$, $i = 1, \dots, n$.

Obviously $\alpha = (\alpha_1, \dots, \alpha_n)$ is a fixed point for this functions, i.e. $\varphi(\alpha) = \alpha$. It is not difficult to verify that:

$\dfrac{\partial \varphi_i(\alpha)}{\partial x_j} = 0$ is valid for all $i, j = 1, \dots, n$,

but not fulfilled $\dfrac{\partial^2 \varphi_i(\alpha)}{\partial x_i \partial x_j} = 0$ for all $i, j = 1, \dots, n$. Theorem 3.1 implies that the iteration (2.12) converges with second order of convergence.

3.2 Second modification

In this subsection we analyze the local convergence property of *modified inverse WDK method* (2.13). The following theorem deals with the order of the convergence of the method.

Theorem 3.2 *Let consider a monic polynomial p(x) of degree n defined by (1.1), with simple zeros* $\alpha_1, \alpha_2, \dots, \alpha_n$. *Then for sufficiently close distinct initial approximations* $x_1^0, \dots, x_n^0$ *of the zeros respectively the iterative method (2.13) is convergent to the zeros with second order of convergence.*

Proof. For the sake of brevity we denote:

$\hat{x}_i = x_i^{k+1}$, $x_i = x_i^k$, $\hat{\varepsilon}_i = x_i^{k+1} - \alpha_i$, $\varepsilon_i = x_i^k - \alpha_i$ and $W_i = W_i(x^k)$.

In another description formula (k2.13) is

$$\hat{x}_i = x_i - \frac{W_i}{1 + \dfrac{W_i}{x_i}}.$$

Thus we get the expression

$$\hat{\varepsilon}_i = \varepsilon_i \left[1 - \frac{\prod_{j \neq i}^{n} \dfrac{x_i - \alpha_j}{x_i - x_j}}{1 + \dfrac{W_i}{x_i}} \right] = \varepsilon_i \left[\frac{1 - \prod_{j \neq i}^{n} \dfrac{x_i - \alpha_j}{x_i - x_j} + \dfrac{W_i}{x_i}}{1 + \dfrac{W_i}{x_i}} \right].$$

The following correlation is known in the literature

$$\prod_{j \neq i}^{n} \frac{x_i - \alpha_j}{x_i - x_j} - 1 = \sum_{k \neq i}^{n} \frac{\varepsilon_k}{x_i - x_k} \prod_{j \neq i}^{k-1} \frac{x_i - \alpha_k}{x_i - x_j}.$$

Then after some expressions we get

$$\hat{\varepsilon}_i = \varepsilon_i \left[\frac{\dfrac{\varepsilon_i}{x_i} \prod_{j \neq i}^{n} \dfrac{x_i - \alpha_j}{x_i - x_j} - \sum_{k \neq i}^{n} \dfrac{\varepsilon_k}{x_i - x_k} \prod_{j \neq i}^{k-1} \dfrac{x_i - \alpha_k}{x_i - x_j}}{1 + \dfrac{\varepsilon_i}{x_i} \prod_{j \neq i}^{n} \dfrac{x_i - \alpha_j}{x_i - x_j}} \right].$$

If we adopt that the absolute values of all the errors ε_i for $j = 1, \ldots, n$ are of the same order, say $|\varepsilon_j| = O(|\varepsilon|)$, then we obtain the expression

$$|\hat{\varepsilon}_i| = |\varepsilon|^2 \left| \frac{\dfrac{1}{x_i} \prod_{j \neq i}^{n} \dfrac{x_i - \alpha_j}{x_i - x_j} - \sum_{k \neq i}^{n} \dfrac{1}{x_i - x_k} \prod_{j \neq i}^{k-1} \dfrac{x_i - \alpha_k}{x_i - x_j}}{1 + \dfrac{\varepsilon_i}{x_i} \prod_{j \neq i}^{n} \dfrac{x_i - \alpha_j}{x_i - x_j}} \right| = O\left(|\varepsilon|^2\right),$$

which proves the theorem.

4. INVERSE NEWTON ITERATIVE METHOD

It is known that the Weierstrass iterative function (1.2) is a generalization of the known Newton iterative method

(4.1) $x^{k+1} = x^k - u\left(x^k\right),$ for $k = 0,1,2,\ldots,$

where $u\left(x^k\right) = \dfrac{f(x^k)}{f'(x^k)}$ and f is a nonlinear function.

Now let us consider the corresponding method of (2.13) in the one dimensional case

(4.2) $x^{k+1} = \dfrac{x^k}{1 + \dfrac{u(x^k)}{x^k}}$ or $x^{k+1} = x^k - \dfrac{u(x^k)}{1 + \dfrac{u(x^k)}{x^k}},$

where $k = 0,1,2,\ldots$. Further we refer to (4.2) as *Inverse Newton iterative method*.

The following local convergence theorem is valid.

Theorem 4.1 Let $f \in C^3[a,b]$, $f' \neq 0$ *for every* $x \in [a,b]$ *and* α *is a root of the equation* $f(x) = 0$ *located in the interval* (a,b). *Then for sufficiently close initial approximation* $x^0 \in (a,b)$ *the iterative method (4.2) is convergent to* α *with second order of convergence.*

Proof. For the sake of brevity denote $x = x^k$. Let denote the right side of (4.2) with

$$\varphi(x) = \frac{x}{1 + \dfrac{u(x)}{x}}.$$

Using the known expressions

$u(\alpha) = 0$, $u'(\alpha) = \dfrac{f'(\alpha)^2 - f(\alpha)f''(\alpha)}{f'(\alpha)^2} = 1$ and $u''(\alpha) = 0$ (from the convergence analysis of Newton's iterative method), we get that $\varphi(\alpha) = \alpha$, i.e. α is a fixed point for the function $\varphi(x)$.
Then for the first derivative of φ we have

$$\varphi'(x) = \frac{x^2 + 2xu(x) - x^2u'(x)}{(x + u(x))^2},$$

which implies

$$\varphi'(\alpha) = \frac{\alpha^2 + 2\alpha u(\alpha) - \alpha^2 u'(\alpha)}{(\alpha + u(\alpha))^2} = 0.$$

It is not difficult to verify that

$$\varphi''(\alpha) \neq 0,$$

then from Theorem 3.1 (for $n = 1$) it follows that the Inverse Newton iterative method (4.2) is convergent with second order of convergence.

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