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SPECTRAL STABILITY FOR PERIODIC STANDING WAVES OF THE KLEIN-GORDON SYSTEM*

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ABSTRACT: *We consider spectral stability for periodic wave solutions for the coupled Klein-Gordon equation. We find conditions on parameters of the waves which imply stability and instabilty of periodic standing waves. This is achieved via the abstract stability criteria developed by [1]*

KEYWORDS: *periodic standing waves, Klein-Gordon equation*

1 Introduction

Consider the following coupled Klein-Gordon equations

$$
\left| \begin{array}{l} u_{tt} - u_{xx} + u - (|u|^2 + \beta |v|^2)u = 0 \\ \\ v_{tt} - v_{xx} + v - (\beta |u|^2 + |v|^2)v = 0, \end{array} \right. \tag{1}
$$

where u and v are complex valued functions, and β is real parameter. This system arise as a model of the interaction of two fields [10]. System (1) also can be interpreted as a coupled version of the Klein-Gordon equation

$$
u_{tt} - u_{xx} + u - |u|^2 u = 0. \tag{2}
$$

The existence and stability properties of standing wave $u(t,x) = e^{iwt}\varphi(x)$ is an important question both from theoretical and practical point of view. For the classical nonlinear Klein-Gordon equation

$$
u_{tt} - \Delta u + u - |u|^p u = 0 \tag{3}
$$

Shatah [8] proved the orbital stability of standing waves for $d \geq 3$ and $1 < p < 1 + \frac{4}{d}$, $w_{p,d} < |w| < 1$. Sufficient conditions for instability are given in [3, 5, 6]. The orbital instability for $d \geq 3$ and $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ is established in [9]. Complete characterizations of the linear stability for all values of $p > 1$, all dimensions $d \geq 1$ and all values of $w \in (-1, 1)$ is given in [12].

The stability of periodic waves have been studied extensively in the last decade. The existence and orbital stability of periodic standing waves in one dimensional case for the equation (3) was considered in [7]. The stability and instability of dnoidal and cnoidal type periodic standing waves is obtained by using the theory developed in [5, 6].

General abstract framework of spectral stability for second order Hamiltonian systems, recently developed in [1, 11, 12]. In [11, 12] is studied the stability problem of second order in time nonlinear differential equations on the hole line. In these papers, the authors applied the abstract results to the Boussinesq, Klein-Gordon, and Klein-Gordon-Zakharov equations. In [1], the authors developed the instability index theory for quadratic operator pencils. Using the abstract results of [1] in [2] is studied the spectral stability of two parametric periodic traveling-standing wave solution of the equation (3) in one dimensional case.

In this paper we are interested in spectral stability of periodic standing waves. It is well known that the spectra of linearized equation depends on the choice of function space. In the space

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of periodic functions spectrum consists of isolated eigenvalues, while in the space of bounded functions the spectrum is continuous.

Using the theory developed in [1, 11, 12] for the spectral stability of waves for the second order in time nonlinear equations, we give a complete characterization of the linear stability of dnoidal periodic standing waves for the equation (1) with respect to the perturbation of the same period.

The paper is organized as follows. In Section 2, we present the construction of our main object of study - the periodic standing waves. This is not a new material by any means, but we do it in order to single out the solutions of interest. In Section 3, we setup the spectral stability problem and we outline the theory for linearized stability for second order PDE in [1], and point out to the relevant spectral theoretic results about their linearized operators. In Section 4, we prove the main results.

2 Periodic standing waves

In this section we will construct the periodic standings waves for the system (1) in the form $u(t,x)=e^{iwt}\varphi(x),\ v(t,x)=e^{iwt}\psi(x)$, where φ and ψ are periodic functions with period $2T$. Plugging in the system , we obtain

$$(4)\qquad \left| \begin{array}{l} -\varphi''+\sigma\varphi-(\varphi^2+\beta\psi^2)\varphi=0 \\ \\ -\psi''+\sigma\psi-(\beta\varphi^2+\psi^2)\psi=0, \end{array} \right.$$

We now looking for periodic standing wave solutions of (4) in two cases of interest.

2.1 case $(\varphi,0)$

In this case we look for periodic waves for the equation (4) when $\psi=0$. Now equation (4) is reduced to the equation

$$(5)\qquad -\varphi''+(1-w^2)\varphi-\varphi^3=0.$$

Integrating once the above equation, we get

$$(6)\qquad \varphi'^2=-\frac{1}{2}\varphi^4+\sigma\varphi^2+\frac{a}{2},$$

or

$$(7)\qquad \varphi'^2=\frac{1}{2}\left(-\varphi^4+2\sigma\varphi^2+a\right),$$

where $\sigma=1-w^2$ and a is a constant of integration. Let $a<0$ and $\sigma>0$. Denote by $\varphi_0>\varphi_1>0$ the positive solutions of $-\rho^4+2\sigma\rho^2+a=0$. Then $\varphi_1\le\varphi\le\varphi_0$ and the solution φ

$$(8)\qquad \varphi(x)=\varphi_0 dn(\alpha x,\kappa),$$

where

$$(9)\qquad \varphi_0^2+\varphi_1^2=2\sigma,\ \ \alpha=\sqrt{\frac{1}{2}}\varphi_0,\ \ \kappa^2=\frac{\varphi_0^2-\varphi_1^2}{\varphi_0^2}=\frac{2\varphi_0^2-2\sigma}{\varphi_0^2}.$$

Since dn has a fundamental period $2K(\kappa)$, then the fundamental period of solution (8) is

(10) $$2T = \frac{2K(\kappa)}{\alpha}, \ T \in I = \left(\frac{\sqrt{2}\pi}{\sqrt{\sigma}}, \infty \right).$$

Here and below $K(k)$ and $E(k)$ are, as usual, the complete elliptic integrals of the first and second kind in a Legendre form.

2.2 case (φ, φ)

We consider the case $\varphi = \psi$. For the periodic function φ, we have the following ODE

(11) $$-\varphi'' + \sigma\varphi - (\beta+1)\varphi^3 = 0.$$

Multiplying the equation (11) by φ' and integrating, we obtain the equation

(12) $$\varphi'^2 = \frac{\beta+1}{2}\left[-\varphi^4 + \frac{2\sigma}{\beta+1}\varphi^2 + a \right],$$

where a is a constant of integration. Let $\varphi_0 > \varphi_1 > 0$ are positive roots of the polynomial $-\rho^4 + \frac{2\sigma}{\beta+1}\rho^2 + a$. Then up to translation the solution of (12) is given by

(13) $$\varphi(x) = \varphi_0 dn(\alpha x, \kappa),$$

where

(14) $$\varphi_0^2 + \varphi_1^2 = \frac{2\sigma}{\beta+1}, \ \alpha = \sqrt{\frac{\beta+1}{2}}\varphi_0, \ \kappa^2 = \frac{\varphi_0^2 - \varphi_1^2}{\varphi_0^2}.$$

3 Linearized Equation

In this section, we set up the linearized problem for (1). We show that it reduced to the study of quadratic operator pencil, for which the general theory is developed in [1, 11, 12]. We take the perturbation in the form

$$u(t,x) = e^{iwt}(\varphi(x) + p(t,x)), \ u(t,x) = e^{iwt}(\varphi(x) + q(t,x)),$$

where $p(t,x)$ and $q(t,x)$ are complex valued functions. Plugging this ansatz in (1) and ignoring all quadratic and higher order terms yields the following *linear* equation for (p,q)

(15) $$\begin{cases} p_{tt} + 2iwp_t - p_{xx} + \sigma p - (\beta+1)\varphi^2 p - 2\varphi^2 Rep - 2\beta\varphi^2 Req = 0 \\ q_{tt} + 2iwq_t - q_{xx} + \sigma q - (\beta+1)\varphi^2 q - 2\varphi^2 Req - 2\beta\varphi^2 Rep = 0 \end{cases}$$

Splitting the real and imaginary parts $p = F + iG, \ q = R + iS$ allows us to rewrite the linearized problem as the following system

(16) $$\vec{U}_{tt} + J\vec{U}_t + \mathscr{H}\vec{U} = 0,$$

where $\vec{U}=(F,R,G,S)$ and

$$J=\begin{pmatrix} 0 & 0 & -2w & 0 \\ 0 & 0 & 0 & -2w \\ 2w & 0 & 0 & 0 \\ 0 & 2w & 0 & 0 \end{pmatrix} \quad \mathscr{H}=\begin{pmatrix} H_1 & -2\beta\varphi\psi & 0 & 0 \\ -2\beta\varphi\psi & H_2 & 0 & 0 \\ 0 & 0 & H_3 & 0 \\ 0 & 0 & 0 & H_4 \end{pmatrix},$$

$$\begin{aligned} H_1 &= -\partial_x^2+\sigma-3\varphi^2-\beta\psi^2 \\ H_2 &= -\partial_x^2+\sigma-\beta\varphi^2-3\psi^2 \\ H_3 &= -\partial_x^2+\sigma-\varphi^2-\beta\psi^2 \\ H_4 &= -\partial_x^2+\sigma-\beta\varphi^2-\psi^2. \end{aligned}$$

Note that $\mathscr{H}^*=\mathscr{H}$, while $J^*=-J$. If we consider the eigenvalue problem associated with (16), that is $\vec{U}=e^{\lambda t}\vec{V}$, we arrive at

$$\lambda^2\vec{V}+\lambda J\vec{V}+\mathscr{H}\vec{V}=0 \tag{17}$$

Definition 1. *We say that the quadratic pencil given by the couple (J,H) is scpectrally unstable, if there exists an T periodic function $\vec{V}\in D(\mathscr{H})$ and $\lambda:\Re\lambda>0$, so that*

$$\lambda^2\vec{V}+\lambda J\vec{V}+H\vec{V}=0.$$

Otherwise, we say that the quadratic pencil (J,H) is stable.

3.1 Stability of quadratic pencils

We now present the theory developed in [1] for the stability of quadratic pencils, which we will be able to apply to our problem (17). We only present a corollary of the results therein, which fits our purposes. We start with some notations.

For a self-adjoint operator H, define the number of the negative eigenvalues

$$n(H)=\#\{\lambda\in(-\infty,0)\cap\sigma(H)$$

Next, for a subspace M denote $P_M:M\to M$ to be the orthogonal projection onto the subspace M and $P_M^\perp:=Id-P_M$. For an operator T, denote $T_M:=P_MTP_M$. In particular, if M is finite dimensional, with orthogonal basis $\{\chi_j\}_{j=1}^n$, P_M is given in a matrix form by $\{\langle T\chi_i,\chi_j\rangle\}_{i,j=1}^n$.

Theorem 1. *Let $H=H^*$, so that $n(H)<\infty$ and $dim(Ker(H))<\infty$. Assume that*

1. $$(P_H^\perp HP_H^\perp)^{-1},(P_H^\perp HP_H^\perp)^{-1}P_H^\perp JP_H^\perp, \tag{18}$$

   are both compact operators on L^2.
2. *There exists a positive operator S, so that $X_S=\{u:\langle Su,Su\rangle<\infty\}$ is dense in L^2.*
3. *$S^{-1}JS^{-1},S^{-1}$ are compact, whereas $S(H+\lambda)^{-1}S$ is compact for $\lambda>>1$.*
4. *$J:Ker(H)\to Ker(H)^\perp$*

5. $(Id - JH^{-1}J)|_{Ker(H)}$ *is invertible*[†]

Then,

$$k_r + k_c + k_- = n(H) - n((Id - JH^{-1}J)|_{Ker(H)}). \tag{19}$$

where k_r *is the number of positive solutions* λ *of* (17)*,* k_- *is the number of solutions* λ *of* (17) *with positive real part, whereas* k_c *is the total Krein index for the quadratic pencil.*

Remark: The non-negative integers k_c, k_- are even. As a consequence, if $n(H) = 1$, it follows from (19) that $k_c = k_- = 0$ and

$$k_r = 1 - n((Id - JH^{-1}J)|_{Ker(H)}). \tag{20}$$

If the right side of (19) is odd number, then $k_r \geq 1$ and hence we have instability.

4 Main Results.

We are interested in the spectral stability of periodic standing waves for the system (1). We apply the theory developed in [1] for the stability of quadratic operator pencils. We review and state main results regarding the spectral theory of Hill operators arising in the linearization around corresponding standing waves. First, we establish the spectral stability of periodic standing wave solutions $(\varphi, 0)$.

Theorem 2. *Let* $0 < \beta < 1$*. The dnoidal solution* $(\varphi, 0)$*, where* φ *described in (8), is scpectrally stable for* $|w| > \sqrt{\frac{1}{1+M(\kappa)}}$*, and unstable for* $|w| < \sqrt{\frac{1}{1+M(\kappa)}}$ *where*

$$M(\kappa) := \frac{(2-\kappa^2)[E^2(\kappa) - (1-\kappa^2)K^2(\kappa)]}{(2-\kappa^2)E^2(\kappa) - 2(1-\kappa^2)E(\kappa)K(\kappa)}.$$

For $1 < \beta < 3$ *and* $|w| > \sqrt{\frac{1}{1+M(\kappa)}}$*, the solution are unstable.*

We now give a different formulation of the main result. Let $T > \sqrt{2}\pi$. Then, the waves described in (8) are one parameter family of waves, having a fundamental period $2T$, which can be parametrized by $w : |w| < \sqrt{1 - \frac{2\pi^2}{T^2}}$, (note that w, κ are in one-to-one relation given by (10)). Now, Theorem 2 asserts that the stable waves in this family are exactly those with $|w| \geq w_T$, where $w_T \in \left(0, \sqrt{1 - \frac{2\pi^2}{T^2}}\right)$ is determined as follows. Let κ_T be the unique solution of

$$\left(1 - \frac{K(\kappa)(2-\kappa^2)}{T^2}\right)(1 + M(\kappa)) = T$$

Then, $w_T = \sqrt{\frac{1}{1+M(\kappa_T)}}$.

Proof of Theorem 2. First, we will verify that the pencil given by $(J, \mathscr{H})$ (or equivalently the eigenvalue problem (17)) satisfy the requirements 1-3 of Theorem 1. The compactness of the

[†] Note that this is well-defined since $J : Ker(H) \to Ker(H)^{\perp}$ and hence $JH^{-1}J$ is well-defined.

operators in (18) follows from the compactness of the embeddings $H^2[-L,L] \hookrightarrow H^1[-L,L] \hookrightarrow L^2[-L,L]$. We take the operator

$$S := \begin{pmatrix} (1+|\partial_x|)^{3/4} & 0 & 0 & 0 \\ 0 & (1+|\partial_x|)^{3/4} & 0 & 0 \\ 0 & 0 & (1+|\partial_x|)^{3/4} & 0 \\ 0 & 0 & 0 & (1+|\partial_x|)^{3/4} \end{pmatrix}.$$

Clearly $X_S = H^{3/4}[-L,L] \times H^{3/4}[-L,L] \times H^{3/4}[-L,L] \times H^{3/4}[-L,L]$ is dense in $L^2 \times L^2 \times L^2 \times L^2$. In addition, $S^{-1}JS^{-1}$ is smoothing of order $1/2$, S^{-1} smoothing of order $3/2$, whereas $S(H+\lambda)^{-1}S, \lambda >> 1$ is smoothing of order $1/2$. Thus, all the operators in question are compact in their action on $L^2 \times L^2 \times L^2 \times L^2$.

Next, we will consider the spectral properties of the operator of linearization. Note that in this case the operator of linearization $\mathscr{H}$ is in the form

$$\mathscr{H} = \begin{pmatrix} Q_1 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & 0 \\ 0 & 0 & Q_3 & 0 \\ 0 & 0 & 0 & Q_2 \end{pmatrix},$$

where

$$\begin{aligned} Q_1 &= -\partial_x^2 + \sigma - 3\varphi^2 \\ Q_2 &= -\partial_x^2 + \sigma - \beta\varphi^2 \\ Q_3 &= -\partial_x^2 + \sigma - \varphi^2. \end{aligned}$$

These operators are self-adjoint acting on $L^2_{per}[0,2T]$ with domain $H^2_{per}[0,2T]$. From the Floquet theory, it follows that its spectrum is purely discrete

$$\nu_0 < \nu_1 \le \nu_2 < \nu_3 \le \nu_4 < \dots$$

where ν_0 is always a simple eigenvalue. If $\phi_n(x)$ is the eigenfunction corresponding to ν_n, then

$$\begin{aligned} &\phi_0 \text{ has no zeroes in } [0,2T]; \\ &\phi_{2n+1},\ \phi_{2n+2} \text{ have each just } 2n+2 \text{ zeroes in } [0,2T). \end{aligned}$$

Using (8) and (9), and taking $y = \alpha x$ as an independent variable in Q_1, one obtains $Q_1 = \alpha^2\Lambda_1$ with an operator Λ_1 in $[0,2K(k)]$ given by

$$\Lambda_1 = -\frac{d^2}{dy^2} + 6k^2 sn^2(y;k) - 4 - \kappa^2,$$

where we have used the relation $dn^2(x;\kappa) + \kappa^2 sn^2(x,\kappa) = 1$. The operator Λ_1 is Hill's operator with Lamé potential. It is well-known that the first five eigenvalues of $\Lambda_2 = -\partial_y^2 + 6k^2 sn^2(y,k)$, with periodic boundary conditions on $[0,4K(k)]$ are simple. These eigenvalues and corresponding eigenfunctions are:

$$\begin{aligned} &\nu_0 = 2 + 2k^2 - 2\sqrt{1-k^2+k^4}, && \phi_0(y) = 1 - (1 + k^2 - \sqrt{1-k^2+k^4})sn^2(y,k), \\ &\nu_1 = 1 + k^2, && \phi_1(y) = cn(y,k)dn(y,k) = sn'(y,k), \\ &\nu_2 = 1 + 4k^2, && \phi_2(y) = sn(y,k)dn(y,k) = -cn'(y,k), \\ &\nu_3 = 4 + k^2, && \phi_3(y) = sn(y,k)cn(y,k) = -k^{-2}dn'(y,k), \\ &\nu_4 = 2 + 2k^2 + 2\sqrt{1-k^2+k^4}, && \phi_4(y) = 1 - (1 + k^2 + \sqrt{1-k^2+k^4})sn^2(y,k). \end{aligned}$$

Since the eigenvalues of Q_1 and Λ_1 are related by $\lambda_n = \alpha^2 \nu_n$, it follows that the first three eigenvalues of the operator Q_1, equipped with periodic boundary condition on $[0,2K(k)]$ are simple and $\lambda_0 < 0, \lambda_1 = 0, \lambda_2 > 0$. The corresponding eigenfunctions are $\phi_0(\alpha x), \phi_1(\alpha x) = const.\varphi'$ and $\phi_2(\alpha x)$.

From (11), we have $Q_3\varphi = 0$ and since $\varphi > 0$ it follows that zero is the first eigenvalue of operator Q_3, which is simple.

First we consider the case $0 < \beta < 1$. We have $Q_2 > Q_3$, and therefore operator Q_2 is strong positive and $KerQ_2 = \{\varnothing\}$.

Hence, $n(\mathscr{H}) = 1$, $dimKer\mathscr{H} = 2$, and $\vec{\psi}_0 = \frac{1}{||\varphi||}(0,0,\varphi,0)$, $\vec{\psi}_1 = \frac{1}{||\varphi'||}(\varphi',0,0,0) \in Ker\mathscr{H}$.

The anti-selfadjointness of J yields $\langle J\psi_i, \psi_i\rangle = 0, i = 1,2$. By direct computations $\langle J\psi_i, \psi_j\rangle = 0$, $i,j = 1,2$. With this, we have verified that $J : Ker(\mathscr{H}) \to Ker(\mathscr{H})^{\perp}$. Thus, in order to determine the stability one needs to compute the index $n((Id - J\mathscr{H}^{-1}J)|_{Ker(\mathscr{H})})$. For the $(Id - J\mathscr{H}^{-1}J)|_{Ker(\mathscr{H})}$ we have the following matrix representation

$$U := \begin{pmatrix} \langle (Id - J\mathscr{H}^{-1}J)\vec{\psi}_0, \psi_0\rangle & \langle (Id - J\mathscr{H}^{-1}J)\vec{\psi}_0, \psi_1\rangle \\ \langle (Id - J\mathscr{H}^{-1}J)\vec{\psi}_1, \psi_0\rangle & \langle (Id - J\mathscr{H}^{-1}J)\vec{\psi}_1, \psi_1\rangle \end{pmatrix}$$

We have

$$\mathscr{H}^{-1}J\vec{\psi}_0 = \frac{1}{||\varphi||}\begin{pmatrix} -2wQ_1^{-1}\varphi \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathscr{H}^{-1}J\vec{\psi}_1 = \frac{1}{||\varphi'||}\begin{pmatrix} 0 \\ 0 \\ Q_3^{-1}\varphi' \\ 0 \end{pmatrix}.$$

Thus

$$U = \begin{pmatrix} 1 + \frac{4w^2}{||\varphi||^2}\langle Q_1^{-1}\varphi, \varphi\rangle & 0 \\ 0 & 1 + \frac{4w^2}{||\varphi'||^2}\langle Q_3^{-1}\varphi', \varphi'\rangle \end{pmatrix}.$$

Since the operator Q_3 is non-negative, then $\langle Q_3^{-1}\varphi', \varphi'\rangle \geq 0$.

Hence, the stability criteria is as follows

- Stability, if $1 + \frac{4w^2}{||\varphi||^2}\langle Q_1^{-1}\varphi, \varphi\rangle < 0$ (one negative and positive eigenvalues of U, $n((Id - J\mathscr{H}^{-1}J)|_{Ker(\mathscr{H})}) = 1$)

-Instability, if $1 + \frac{4w^2}{||\varphi||^2}\langle Q_1^{-1}\varphi, \varphi\rangle > 0$ (two positive eigenvalues of U, $n((Id - J\mathscr{H}^{-1}J)|_{Ker(\mathscr{H})}) = 0$)

Thus it remains to compute $\langle Q_1^{-1}\varphi, \varphi\rangle$. We have $Q_1\varphi' = 0$. The function

$$\psi(x) = \varphi'(x)\int^x \frac{1}{\varphi'^2(s)}ds, \quad \begin{vmatrix} \varphi' & \psi \\ \varphi'' & \psi' \end{vmatrix} = 1$$

is also solution of $Q_1\psi = 0$. Formally, since φ' has zeros using the identities

$$\frac{1}{cn^2(y,\kappa)} = \frac{1}{dn(y,\kappa)}\frac{\partial}{\partial_y}\frac{sn(x,\kappa)}{cn(y,\kappa)}, \quad \frac{1}{sn^2(y,\kappa)} = -\frac{1}{dn(y,\kappa)}\frac{\partial}{\partial_y}\frac{cn(x,\kappa)}{sn(y,\kappa)}$$

and integrating by parts we get

$$\psi(x) = \frac{1}{\alpha^2\kappa^2\varphi_0}\left[\frac{1-2sn^2(\alpha x,\kappa)}{dn(\alpha x,\kappa)} - \alpha\kappa^2 sn(\alpha x,\kappa)cn(\alpha x,\kappa)\int_0^x \frac{1-2sn^2(\alpha s,\kappa)}{dn^2(\alpha s,\kappa)}ds\right]$$

Thus, we may construct Green function

$$Q_1^{-1}f=\varphi'\int_0^x \psi(s)f(s)ds-\psi(s)\int_0^x \varphi'(s)f(s)s+C_f\psi(x),$$

where C_f is chosen such that $Q_1^{-1}f$ is periodic with same period as $\varphi(x)$.

After integrating by parts, we get

$$\langle Q_1^{-1}\varphi,\varphi\rangle=-\langle\varphi^3,\psi\rangle+\frac{\varphi^2(T)+\varphi(0)^2}{2}\langle\varphi,\psi\rangle+C_\varphi\langle\varphi,\psi\rangle, \tag{21}$$

We have

$$\begin{aligned}
&\langle\varphi,\psi\rangle=\tfrac{1}{\alpha^3\kappa^2}[E(\kappa)-K(\kappa)]\\
&\langle\varphi^3,\psi\rangle=\tfrac{\varphi_0^2}{2\alpha^3\kappa^2}[(2-\kappa^2)E(\kappa)-2(1-\kappa^2)K(\kappa)]\\
&C_\varphi=-\tfrac{\varphi''(T)}{2\psi'(T)}\langle\varphi,\psi\rangle+\tfrac{\varphi^2(T)-\varphi^2(0)}{2}.
\end{aligned} \tag{22}$$

With this finally we get

$$\langle Q_1^{-1}\varphi,\varphi\rangle=\frac{\varphi_0^2}{2\alpha^3\kappa^2}\frac{E^2(\kappa)-(1-\kappa^2)K^2(\kappa)}{2(1-\kappa^2)K(\kappa)-(2-\kappa^2)E(\kappa)}<0.$$

We have

$$||\varphi||^2=\int_0^T \varphi_0^2dn^2(\alpha x,\kappa)dx=\frac{2\varphi_0^2}{\alpha}\int_0^{K(\kappa)}dn^2(y,\kappa)dy=\frac{2\varphi_0^2}{\alpha}E(\kappa)$$

Hence

$$1+\frac{4w^2}{||\varphi||^2}\langle Q_1^{-1}\varphi,\varphi\rangle=1-\frac{w^2}{1-w^2}\frac{(2-\kappa^2)[E^2(\kappa)-(1-\kappa^2)K^2(\kappa)]}{(2-\kappa^2)E^2(\kappa)-2(1-\kappa^2)E(\kappa)K(\kappa)}. \tag{23}$$

Thus, the standing waves are stable if $|w|>\frac{1}{\sqrt{(1+M(\kappa)}}$ and unstable if $|w|<\frac{1}{\sqrt{(1+M(\kappa)}}$.

Now if $1<\beta<3$, then $Q_1<Q_2<Q_3$. From Comparison Theorem, $n(Q_2)=1$ and $KerQ_2=\{\emptyset\}$. Hence, $n(\mathscr{H})=2$ and for $|w|<\frac{1}{\sqrt{(1+M(\kappa)}}$, $n(\mathscr{H})-n(U)=1$. □

Now we will consider the spectral stability of periodic standing waves of the form (φ,φ), where φ is given by (13). We have the following result.

Theorem 3. *Let $\beta>1$. The two parameter family of dnoidal solutions (φ,φ), where φ described in (13), are spectrally stable if $|w|>\sqrt{\frac{1}{1+M(\kappa)}}$, and unstable if $|w|<\sqrt{\frac{1}{1+M(\kappa)}}$, where*

$$M(\kappa):=\frac{(2-\kappa^2)[E^2(\kappa)-(1-\kappa^2)K^2(\kappa)]}{(2-\kappa^2)E^2(\kappa)-2(1-\kappa^2)E(\kappa)K(\kappa)}.$$

For $0<\beta<1$ and $|w|>\sqrt{\frac{1}{1+M(\kappa)}}$, the solutions (φ,φ) are unstable.

Proof. Similarly as in the Theorem 2, we have that the pencil given by $(J,\mathscr{H})$ satisfy the requirements 1-3 of Theorem 1.

Note that in the case $\varphi=\psi$ the operator $\mathscr{H}$ can be diagonalized. Let

$$B=\begin{pmatrix} 1 & 1 & 0 & 0\\ -\frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Then

$$\mathscr{H}_D=B\mathscr{H}B^{-1}=\begin{pmatrix} L_1 & 0 & 0 & 0\\ 0 & L_2 & 0 & 0\\ 0 & 0 & L_3 & 0\\ 0 & 0 & 0 & L_3\end{pmatrix},$$

where

$$\begin{aligned} L_1&=-\partial_x^2+\sigma-3(\beta+1)\varphi^2\\ L_2&=-\partial_x^2+\sigma-(3-\beta)\varphi^2\\ L_3&=-\partial_x^2+\sigma-(\beta+1)\varphi^2.\end{aligned}$$

Next, we list the spectral properties of the operator $\mathscr{H}$ and $\mathscr{H}_D$. Since $\mathscr{H}_D$ is a diagonal operator, with entries $L_i,\ i=1,2,3$, we have that $\sigma(\mathscr{H}_D)=\sigma(L_1)\cup\sigma(L_2)\cup\sigma(L_3)$. The operators $L_i,\ i=1,2,3$ are clearly Hill operators, so that the standard Floquet theory is applicable to them.

Using that $\kappa^2 sn^2(y)+dn^2(y)=1$ and formulas (14), we obtain

$$\begin{aligned} L_1\ &=-\partial_x^2+\sigma-3(\beta+1)\varphi_0^2dn^2(\alpha x,\kappa)\\ &=\alpha^2\left[-\partial_y^2+6\kappa^2sn^2(y,\kappa)-(4+\kappa^2)\right],\end{aligned}$$

where $y=\alpha x$.

It follows that the first three eigenvalues of the operator L_1, equipped with periodic boundary condition on $[0,2K(k)]$ are simple.

From (11), we have that $L_3\varphi=0$ and since $\varphi>0$, then zero is the first eigenvalue, which is simple with corresponding eigenfunction φ. Moreover, $L_2=L_3+2(\beta-1)\varphi^2$ and for $\beta>1$, the operator L_2 is strong positive and $KerL_2=\{\emptyset\}$.

Hence, $n(\mathscr{H})=1$, $dimKer\mathscr{H}=3$, and $\Psi_0=\frac{1}{\sqrt{2}||\varphi'||}(\varphi',\varphi',0,0),\Psi_1=\frac{1}{||\varphi||}(0,0,\varphi,0),\Psi_2=\frac{1}{||\varphi||}(0,0,0,\varphi)\in Ker\mathscr{H}$.

The anti-selfadjointness of J yields $\langle J\psi_i,\psi_i\rangle=0,i=1,2$. Moreover, by direct computations $\langle J\psi_i,\psi_j\rangle=0,\ i,j=1,2$. With this, we have verified that $J:Ker(\mathscr{H})\to Ker(\mathscr{H})^{\perp}$.

No we will estimate the index $n((Id-J\mathscr{H}^{-1}J)|_{Ker(H)})$. For the $(Id-J\mathscr{H}^{-1}J)|_{Ker(H)}$ we have the following matrix representation

$$U:=\begin{pmatrix} \langle(Id-J\mathscr{H}^{-1}J)\Psi_0,\Psi_0\rangle & \langle(Id-J\mathscr{H}^{-1}J)\Psi_0,\Psi_1\rangle & \langle(Id-J\mathscr{H}^{-1}J)\Psi_0,\Psi_2\rangle\\ \langle(Id-J\mathscr{H}^{-1}J)\Psi_1,\Psi_0\rangle & \langle(Id-J\mathscr{H}^{-1}J)\Psi_1,\Psi_1\rangle & \langle(Id-J\mathscr{H}^{-1}J)\Psi_1,\Psi_2\rangle\\ \langle(Id-J\mathscr{H}^{-1}J)\Psi_2,\Psi_0\rangle & \langle(Id-J\mathscr{H}^{-1}J)\Psi_2,\Psi_1\rangle & \langle(Id-J\mathscr{H}^{-1}J)\Psi_2,\Psi_2\rangle\end{pmatrix}$$

We have

$$\mathscr{H}^{-1}J\Psi_0=\frac{\sqrt{2}w}{||\varphi||}\begin{pmatrix}0\\0\\L_3^{-1}\varphi'\\L_3^{-1}\varphi'\end{pmatrix}$$

$$\mathscr{H}^{-1}J\Psi_1 = -\frac{w}{||\varphi'||}\begin{pmatrix} L_1^{-1}\varphi + L_2^{-1}\varphi \\ L_1^{-1}\varphi - L_2^{-1}\varphi \\ 0 \\ 0 \end{pmatrix},$$

$$\mathscr{H}^{-1}J\Psi_2 = -\frac{w}{||\varphi||}\begin{pmatrix} L_1^{-1}\varphi - L_2^{-1}\varphi \\ L_1^{-1}\varphi + L_2^{-1}\varphi \\ 0 \\ 0 \end{pmatrix}.$$

Thus

$$U = \begin{pmatrix} 1+C & 0 & 0 \\ 0 & 1+A+B & A-B \\ 0 & A-B & 1+A+B \end{pmatrix},$$

where

$$A = \frac{2w^2}{||\varphi||^2}\langle L_1^{-1}\varphi, \varphi\rangle,\ B = \frac{2w^2}{||\varphi||^2}\langle L_2^{-1}\varphi, \varphi\rangle,\ C = \frac{2w^2}{||\varphi'||^2}\langle L_3^{-1}\varphi', \varphi'\rangle$$

The eigenvalues of the matrix U are $\rho_1 = 1+2A$, $\rho_2 = 1+2B$, $\rho_3 = 1+C$. Since, L_2 is non-negative and L_3 is strong positive, then $\rho_2 = 1+2B = 1+\frac{4w^2}{||\varphi||^2}\langle L_2^{-1}\varphi, \varphi\rangle > 0$ and $\rho_3 = 1+C = 1+\frac{2w^2}{||\varphi'||^2}\langle L_3^{-1}\varphi', \varphi'\rangle > 0$.

Hence, the stability criteria is as follows

- Stability, if $1+2A = 1+\frac{4w^2}{||\varphi||^2}\langle L_1^{-1}\varphi, \varphi\rangle < 0$ (one negative and tow positives eigenvalues of U, $n((Id - J\mathscr{H}^{-1}J)|_{Ker(\mathscr{H})}) = 1$)

-Instability, if $1+2A = 1+\frac{4w^2}{||\varphi||^2}\langle L_1^{-1}\varphi, \varphi\rangle > 0$ (three positive eigenvalues of U, $n((Id - J\mathscr{H}^{-1}J)|_{Ker(\mathscr{H})}) = 0$)

Similarly as in the previous case, we get

$$\rho_1 = 1 - \frac{w^2}{1-w^2}\frac{(2-\kappa^2)E^2(\kappa) - (1-\kappa^2)(2-\kappa^2)K^2(\kappa)}{(2-\kappa^2)E^2(\kappa) - 2(1-\kappa^2)E(\kappa)K(\kappa)}.$$

Thus, the standing waves are stable if $|w| > \sqrt{\frac{1}{1+M(\kappa)}}$ and unstable if $|w| < \sqrt{\frac{1}{1+M(\kappa)}}$.

Now if $0 < \beta < 1$, then $L_1 < L_2 < L_3$. From the Comparison Theorem, we have $n(L_2) = 1$ and $KerL_2 = \{\emptyset\}$, and $n(\mathscr{H}) - n(U) = 1$. $\square$

REFERENCES:

[1] J. Bronski, M. Johnson, T. Kapitula, *An instability index theory for quadratic pencils and applications*, to appear, Comm. Math. Phys. (2014).

[2] A. Demirkaya, S. Hakkaev, M. Stanislavova, A. Stefanov, *On the spectral stability of periodic waves of the Klein-Gordon equation*, Differential and Integral Equations, **28** (2015), 431-454

[3] M. Grillakis, *Linearized instability for nonlinear Schrodinger and Klein-Gordon equations*, Comm. Pure Appl. Math., **41**(1988), no.6, 747-774.

[4] M. Grillakis, *Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system*, Comm. Pure Appl. Math. **43** (1990), p. 299–333.

[5] M. Grillakis, J. Shatah, W. Strauss, *Stability of solitary waves in the presence of symmetry* I, J. Funct. Anal. **74** (1987), 160–197.

[6] M. Grillakis, J. Shatah, W. Strauss, *Stability of solitary waves in the presence of symmetry* II, J. Funct. Anal. **94** (1990), 308–348.

[7] F. Natali, A. Pastor, *Stability and instability of periodic standing wave solutions for some Klein-Gordon equations*, J. Math. Anal. Appl., **347**(2008), p. 428-441

[8] J. Shatah, *Stable standing waves of nonlinear Klein-Gordon equations*, Comm. Math. Phys., **91**(1983), no.3, p. 313–327.

[9] J. Shatah, *Unstable ground states on nonlinear Klein-Gordon equations*, Trans. Amer. Math. Soc., **290**(1985), no.2, 701–710.

[10] I. Segal, *Nonlinear partial differential equations in Quantum Field Theory*, Pros. Symp. Appl. Math. A.M.S. **17**(1965), 210-226

[11] M. Stanislavova, A. Stefanov, *Linear stability analysis for traveling waves of second order in time PDE'S*, Nonlinearity, **25** (2012), p. 2625–2654.

[12] M. Stanislavova, A. Stefanov, *Spectral stability analysis for special solutions of second order in time PDE's: the higher dimensional case*, Physica D, **262** (2013), p. 1–13.

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