LIMIT VALUES OF MULTIPLICATIVE INTEGRALS*

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ABSTRACT: *This paper presents obtaining the explicit form of the limit values of special kind of multiplicative integrals. These multiplicative integrals and the explicit form of their limit values have been essentially used in the investigations of the nondissipative unbounded operators in a Hilbert space (presented as a coupling of dissipative and antidissipative operators and with different domains of the operator and its adjoint), their characteristic operator functions, asymptotic behaviour of the corresponding continuous curves.*

KEYWORDS: *Nonselfadjoint operator, dissipative operator, coupling, unbounded operator, multiplicative integral*

In this paper we present properties of special kind of multiplicative integrals which play an important role in the further development of the investigations of nonselfadjoint unbounded operators (with finite dimensional imaginary parts) based on the theory of the characteristic functions and the triangular models of M.S. Livšic. The presented properties of the multiplicative integrals are so-called limit values of multiplicative integrals connected with nonselfadjoint unbounded K^r-operators A in a Hilbert space H with differen domains of A and its adjoint A^* and presented as a regular coupling of dissipative and antidissipative operators with real absolutely continuous spectra. The triangular model of the regular couplings of this class of nonselfadjoint operators has been introduced and investigated by K.P. Kirchev and G.S. Borisova in [2, 3, 4]. In the course of investigations of these operators A (the characteristic operator functions, the resolvent of A, the asymptotic behaviour of the corresponding continuous curves) we

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use the properties of the multiplicative integrals from the form

$$(1)\qquad \overset{\curvearrowright}{\int_a^b} e^{-i\frac{1+\lambda\alpha(v)}{\alpha(v)-\lambda}T(v)dv},$$

($\lambda \in \mathbb{C}$, $\lambda \neq \alpha(v)$ for $v \in [a,b]$), where $\alpha(v)$ is a nondecreasing real function in $[a,b]$, $T(v)$ is measurable nonnegative $m \times m$ matrix function, satisfying the conditions

$$(2)\qquad \int_a^b trT(v)dv < +\infty; \qquad \int_a^b ||T(v)||dv < +\infty.$$

The integral (1) is the multiplicative Stieltjes integral, defined as

$$(3)\qquad \begin{aligned} \overset{\rightarrow}{\int_a^b} e^{f(t)G(t)dt} &= \lim_{\max \triangle\theta_k \to 0} \overset{\rightarrow}{\prod_{k=1}^{n}} e^{f(\tau_k)(E(\theta_k)-E(\theta_{k-1}))} = \\ &= \lim_{\max \triangle\theta_k \to 0} e^{f(\tau_1)(E(\theta_1)-E(\theta_0))} \ldots e^{f(\tau_n)(E(\theta_n)-E(\theta_{n-1}))}, \end{aligned}$$

where

$$E(\theta) = \int_a^\theta G(t)dt$$

and the limit in (3) is taken over all the partitions $a = \theta_0 < \theta_1 < \ldots < \theta_n = b$ of the interval $[a,b]$ and all the choices of intermediate points τ_k such that $\theta_{k-1} \leq \tau_k \leq \theta_k$ $(k = 1, 2, \ldots, n)$, $G(\theta)$ is integrable matrix function on $[a,b]$ and

$$||E(\theta') - E(\theta'')|| \leq |\theta' - \theta''|.$$

Further to avoid the complications of writing we will consider the multiplicative integral (1) in the case when $\alpha(v) = v$. The general case of $\alpha(v) \neq v$ can be considered analogously.

The limits of the multiplicative integrals from the form (in the sense of a strong limit)

$$(4)\qquad \begin{gathered} s-\lim_{\delta\to 0}\overset{b}{\overrightarrow{\int_a}} e^{\frac{-iT(v)}{v-(x\pm i\delta)}dv} = \\ = s-\lim_{\delta\to 0}\overset{x-\delta}{\overrightarrow{\int_a}} e^{\frac{-iT(v)}{v-x}dv} e^{\pm\pi T(x)} \overset{b}{\overrightarrow{\int_{x+\delta}}} e^{\frac{-iT(v)}{v-x}dv} \end{gathered}$$

are used essentially in the case of bounded dissipative operators [6], the case of bounded nonselfadjoint operators, presented as a coupling of dissipative and antidisipative operators [1], the case of unbounded nonselfadjoint operators, presented as a coupling of dissipative and antidissipative operators and with equal domains of the operator and its adjoint [3]. The equality (4) is proved by L.A. Sakhnovich in [6] and it is an analogue for the multiplicative integrals of the well-known Privalov's theorem [5] for the limit values for the integral

$$f(\lambda)=\int_a^b \frac{p(t)}{t-\lambda}dt$$

in the scalar case.

In the case of considered unbounded K^r- operators in the course of obtaining the asymptotics of the corresponding nondissipative curves we essentially use the existence and the form of the limit values of the multiplicative integrals (in the sense of a strong limit) for almost all $x\in[a,b]$

$$(5)\qquad s-\lim_{\delta\to 0}\overset{b}{\overrightarrow{\int_a}} e^{-i\frac{1+(x\pm i\delta)v}{v-(x\pm i\delta)}T(v)dv},\qquad \delta>0.$$

At first we will remind some propositions, obtained in [6], which we will use.

LEMMA 1. *([6]) Let the matrix function $W(\lambda)$ is analytical and uniformly bounded by norm in the upper half-plane. Then for almost all x there exists the limit*

$$W^{+}(x) = s - \lim_{\delta\to 0} W(x+i\delta), \quad \delta > 0.$$

LEMMA 2. *([6]) Let the matrix function $B(t)$ is integrable in $[a,b]$. Then for almost all x in $[a,b]$ the next equality hold*

$$\lim_{\delta\to 0} \frac{1}{2\delta} \int\limits_{x-\delta}^{x+\delta} ||B(t) - B(x)||dt = 0.$$

LEMMA 3. *([6]) Let $m \times m$ matrix functions $(m \leq \infty)$ $B_1(t)$ and $B_2(t)$ are integrable in $[a,b]$ and for almost all t in $[a,b]$ satisfy the inequalities*

$$\frac{B_k(t) - B_k^{*}(t)}{i} \leq 0, \qquad k = 1, 2.$$

Then

$$\left\| \overset{\overrightarrow{b}}{\underset{a}{\int}} e^{-iB_1(t)dt} - \overset{\overrightarrow{b}}{\underset{a}{\int}} e^{-iB_2(t)dt} \right\| \leq \int\limits_a^b ||B_1(t) - B_2(t)||dt.$$

Now we are in a position to prove the next theorem.

THEOREM 4. *Let the matrix function $T(x)$ is integrable and nonnegative in $[a,b]$. Then for almost all $x \in \mathbb{R}$ there exist the limits (5) and they have the form*

$$s - \lim_{\delta\to 0} \overset{\overrightarrow{b}}{\underset{a}{\int}} e^{-i\frac{1+(x\pm i\delta)v}{v-(x\pm i\delta)}T(v)dv} =$$

(6)

$$= s - \lim_{\varepsilon\to 0} \overset{\overrightarrow{x-\varepsilon}}{\underset{a}{\int}} e^{-i\frac{1+vx}{v-x}T(v)dv} e^{\pm\pi(1+x^2)T(x)} \overset{\overrightarrow{b}}{\underset{x+\varepsilon}{\int}} e^{-i\frac{1+vx}{v-x}T(v)dv}$$

($\delta > 0$, $\varepsilon > 0$).

Proof. We consider the multiplicative integral (5) for $\lambda = x \pm i\delta$ ($\delta > 0$), i.e.

$$(7)\quad s - \lim_{\delta \to 0} \overset{\overset{b}{\rightarrow}}{\underset{a}{\int}} e^{-i\frac{1+(x\pm i\delta)v}{v-(x\pm i\delta)}T(v)dv} = \lim_{\max \triangle v_k \to 0} \overset{\overset{n}{\rightarrow}}{\prod_{k=1}} e^{-i\frac{1+(x\pm i\delta)\theta_k}{\theta_k-(x\pm i\delta)}T(\theta_k)\triangle v_k},$$

where $a = v_0 < v_1 < \ldots < v_n = b$, the intermediate points θ_k such that $v_{k-1} \le \theta_k \le v_k$, $\triangle v_k = v_k - v_{k-1}$ $(k = 1, 2, \ldots, n)$.

At first after direct calculations we obtain that

$$(8)\qquad e^{-i\frac{1+(x\pm i\delta)}{v-x\mp i\delta}T(v)} = e^{\pm\frac{\delta(1+v^2)}{(v-x)^2+\delta^2}T(v)-i\frac{v-x+(x(v-x)+\delta^2)v}{(v-x)^2+\delta^2}T(v)}.$$

Now we will mention that

$$(9)\qquad ||e^{-\beta T}|| \le 1$$

for arbitrary $\beta > 0$ and the matrix $T \ge 0$, because the function

$$(e^{-\beta T}f, e^{-\beta T}f)$$

is nonincreasing function of β, $\beta > 0$ (for fixed $f \in \mathbb{C}^m$).

On the other hand the function $(e^{i\gamma T}f, e^{i\gamma T}f)$ of $\gamma \in \mathbb{R}$ (for fixed $f \in \mathbb{C}^m$ when the matrix T is nonnegative) does not depend on γ and hence

$$(10)\qquad ||e^{i\gamma T}f|| = ||f||$$

and consequently

$$(11)\qquad ||e^{i\gamma T}|| \le 1 \quad \forall\gamma \in \mathbb{R}.$$

But $||f|| = ||e^{i\gamma T}f|| \le ||e^{i\gamma T}||.||f||$ and then

$$(12)\qquad ||e^{i\gamma T}|| \ge 1 \quad \forall\gamma \in \mathbb{R}.$$

The inequalities (11) and (12) imply that

$$(13)\qquad ||e^{i\gamma T}|| = 1 \quad \forall\gamma \in \mathbb{R}.$$

Now using (7), (8), (9) and (13) we obtain the next inequality

$$\left\| \overset{b}{\underset{a}{\overrightarrow{\int}}} e^{-i\frac{1+(x-i\delta)v}{v-x+i\delta}T(v)dv} \right\| \leq 1, \qquad \delta > 0,\ x \in \mathbb{R}. \tag{14}$$

Further for arbitrary $\beta \in \mathbb{R}$ and $\gamma > 0$ using (13) we obtain that

$$||e^{(-\beta-i\gamma)T}f|| \leq ||e^{-\beta T}||.||e^{-i\gamma T}|| = ||e^{-\beta T}||. \tag{15}$$

But for arbitrary fixed $f \in \mathbb{C}^m$ it follows that

$$||e^{-\beta T}f|| = ||e^{(-\beta-i\gamma)T}e^{i\gamma T}f|| \leq ||e^{(-\beta-i\gamma)T}||.||f|| \tag{16}$$

and hence

$$||e^{-\beta T}|| \leq ||e^{(-\beta-i\gamma)T}||. \tag{17}$$

The inequalities (15) and (17) imply that

$$||e^{(-\beta-i\gamma)T}|| = ||e^{-\beta T}|| \qquad (\beta, \gamma \in \mathbb{R}). \tag{18}$$

Now for $\delta > 0$ using (7) and (8) we obtain that

$$\left\| \overset{b}{\underset{a}{\overrightarrow{\int}}} e^{-i\frac{1+(x+i\delta)v}{v-x-i\delta)}T(v)dv} \right\| \leq e^{\int_a^b \frac{\delta(1+v^2)}{(v-x)^2+\delta^2}||T(v)||dv}. \tag{19}$$

Then for the inverse multiplicative integrals

$$\left(\overset{b}{\underset{a}{\overrightarrow{\int}}} e^{-i\frac{1+(x\pm i\delta)v}{v-(x\mp i\delta)}T(v)dv} \right)^{-1} = \overset{b}{\underset{a}{\overleftarrow{\int}}} e^{i\frac{1+(x\pm i\delta)v}{v-(x\mp i\delta)}T(v)dv}$$

we obtain

$$\left\| \overset{b}{\underset{a}{\overleftarrow{\int}}} e^{i\frac{1+(x+i\delta)v}{v-(x+i\delta)}T(v)dv} \right\| \leq 1, \tag{20}$$

Then using the inequality (20) we have

$$(21)\qquad \left\| \overset{\rightarrow}{\int_a^b} e^{-i\frac{1+(x+i\delta)v}{v-(x+i\delta)}} T(v)dv \right\| \geq \left\| \overset{\leftarrow}{\int_a^b} e^{i\frac{1+(x+i\delta)v}{v-(x+i\delta)}} T(v)dv \right\|^{-1} \geq 1.$$

The inequality (21) together with (19) implies that

$$(22)\qquad 1 \leq \left\| \overset{\rightarrow}{\int_a^b} e^{-i\frac{1+(x+i\delta)v}{v-(x+i\delta)}} T(v)dv \right\| \leq e^{\int_a^b \frac{\delta(1+v^2)}{(v-x)^2+\delta^2}||T(v)||dv}.$$

The inequalities (14), (20) and Lemma 1 give the existence of the limits

$$(23)\qquad s-\lim_{\delta\to 0} \overset{\rightarrow}{\int_a^b} e^{-i\frac{1+(x-i\delta)v}{v-x+i\delta}} T(v)dv,$$

$$(24)\qquad s-\lim_{\delta\to 0} \left(\overset{\rightarrow}{\int_a^b} e^{-i\frac{1+(x+i\delta)v}{v-(x+i\delta)}} T(v)dv \right)^{-1}$$

for almost all $x \in \mathbb{R}$. Then there exists the limit

$$(25)\qquad s-\lim_{\delta\to 0} \overset{\rightarrow}{\int_a^b} e^{-i\frac{1+(x+i\delta)v}{v-x-i\delta}} T(v)dv.$$

Now we are in a position to prove the equality (6), i.e. to obtain the explicit form of the limit (6). Let us consider the function of ε

$$(26)\qquad \varphi(\varepsilon) = \varepsilon^2 + \sqrt{\varepsilon \int_{x-\varepsilon}^{x+\varepsilon} ||T(v) - T(x)||dv}, \quad \varepsilon > 0$$

for x such that the next equality is true (using Lemma 2)

$$(27)\qquad \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} ||T(v) - T(x)||dv = 0.$$

Then after straightforward calculations we obtain that

$$
\begin{aligned}
&\left\| \overrightarrow{\int\limits_{a}^{b}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}T(v)dv} - \right. \\
&\left. - \overrightarrow{\int\limits_{a}^{x-\varepsilon}} e^{-i\frac{1+vx}{v-x}T(v)dv} e^{\pm\pi(1+x^2)T(x)} \overrightarrow{\int\limits_{x+\varepsilon}^{b}} e^{-i\frac{1+vx}{v-x}T(v)dv} \right\| \leq \\
&\leq \left\| \overrightarrow{\int\limits_{a}^{x-\varepsilon}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}T(v)dv} - \overrightarrow{\int\limits_{a}^{x-\varepsilon}} e^{-i\frac{1+vx}{v-x}T(v)dv} \right\| \cdot \\
&\cdot \left\| \overrightarrow{\int\limits_{x-\varepsilon}^{b}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}T(v)dv} \right\| + \\
&+ \left\| \overrightarrow{\int\limits_{a}^{x-\varepsilon}} e^{-i\frac{1+vx}{v-x}T(v)dv} \right\| \cdot \left\| \overrightarrow{\int\limits_{x-\varepsilon}^{x+\varepsilon}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}T(v)dv} \right\| \cdot \\
&\cdot \left\| \overrightarrow{\int\limits_{x+\varepsilon}^{b}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}T(v)dv} - \overrightarrow{\int\limits_{x+\varepsilon}^{b}} e^{-i\frac{1+vx}{v-x}T(v)dv} \right\| + \\
&+ \left\| \overrightarrow{\int\limits_{a}^{x-\varepsilon}} e^{-i\frac{1+vx}{v-x}T(v)dv} \right\| \cdot \left\| \overrightarrow{\int\limits_{x+\varepsilon}^{b}} e^{-i\frac{1+vx}{v-x}T(v)dv} \right\| \cdot \\
&\cdot \left\| \overrightarrow{\int\limits_{x-\varepsilon}^{x+\varepsilon}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}T(v)dv} - e^{-iT(x)\int\limits_{x-\varepsilon}^{x+\varepsilon}\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}dv} \right\| + \\
&+ \left\| \overrightarrow{\int\limits_{a}^{x-\varepsilon}} e^{-i\frac{1+vx}{v-x}T(v)dv} \right\| \cdot \left\| e^{-iT(x)\int\limits_{x-\varepsilon}^{x+\varepsilon}\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}dv} - e^{-\pi(1+x^2)T(x)} \right\| \cdot \\
&\cdot \left\| \overrightarrow{\int\limits_{x+\varepsilon}^{b}} e^{-i\frac{1+vx}{v-x}T(v)dv} \right\| .
\end{aligned}
$$

In other words using the inequalities (13) and (9) we have

$$\begin{aligned}
&\left\| \overset{b}{\underset{a}{\overrightarrow{\int}}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}T(v)dv} - \right.\\
&\left. - \overset{x-\varepsilon}{\underset{a}{\overrightarrow{\int}}} e^{-i\frac{1+vx}{v-x}T(v)dv} e^{\pm\pi(1+x^2)T(x)} \overset{b}{\underset{x+\varepsilon}{\overrightarrow{\int}}} e^{-i\frac{1+vx}{v-x}T(v)dv} \right\| \leq \\
&\leq \left\| \overset{x-\varepsilon}{\underset{a}{\overrightarrow{\int}}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}T(v)dv} - \overset{x-\varepsilon}{\underset{a}{\overrightarrow{\int}}} e^{-i\frac{1+vx}{v-x}T(v)dv} \right\| \leq \\
(28)\quad &\leq \left\| \overset{b}{\underset{x+\varepsilon}{\overrightarrow{\int}}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}T(v)dv} - \overset{b}{\underset{x+\varepsilon}{\overrightarrow{\int}}} e^{-i\frac{1+vx}{v-x}T(v)dv} \right\| \leq \\
&\leq \left\| \overset{x+\varepsilon}{\underset{x-\varepsilon}{\overrightarrow{\int}}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}T(v)dv} - e^{-iT(x)\int\limits_{x-\varepsilon}^{x+\varepsilon}\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}dv} \right\| \leq \\
&\leq \left\| e^{-iT(x)\int\limits_{x-\varepsilon}^{x+\varepsilon}\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}dv} - e^{-\pi(1+x^2)T(x)} \right\|.
\end{aligned}$$

Now at first we will calculate

$$e^{-iT(x)\int\limits_{x-\varepsilon}^{x+\varepsilon}\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}dv}.$$

We consider the integral

$$\begin{aligned}
(29)\quad &-i\int\limits_{x-\varepsilon}^{x+\varepsilon}\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}dv = -i\int\limits_{x-\varepsilon}^{x+\varepsilon}\frac{v-x}{(v-x)^2+\varphi^2(\varepsilon)}dv- \\
&-i\int\limits_{x-\varepsilon}^{x+\varepsilon}\frac{vx(v-x)-v\varphi^2(\varepsilon)}{(v-x)^2+\varphi^2(\varepsilon)}dv - \int\limits_{x-\varepsilon}^{x+\varepsilon}\frac{(1+v^2)\varphi(\varepsilon)}{(v-x)^2+\varphi^2(\varepsilon)}dv.
\end{aligned}$$

For the first integral on the right hand side of the equality (29) we obtain that

$$\begin{aligned}
(30)\quad &\int\limits_{x-\varepsilon}^{x+\varepsilon}\frac{v-x}{(v-x)^2+\varphi^2(\varepsilon)}dv = \\
&= \tfrac{1}{2}(\ln(\varepsilon^2+\varphi^2(\varepsilon)) - \ln((-\varepsilon^2)+\varphi^2(\varepsilon))) = 0.
\end{aligned}$$

For the second integral on the right side of the equality (29) we have

$$
\begin{aligned}
&\int_{x-\varepsilon}^{x+\varepsilon} \tfrac{vx(v-x)-v\varphi^2(\varepsilon)}{(v-x)^2+\varphi^2(\varepsilon)}dv = \\
&= x\int_{x-\varepsilon}^{x+\varepsilon} dv - x\varphi^2(\varepsilon)\int_{x-\varepsilon}^{x+\varepsilon} \tfrac{1}{(v-x)^2+\varphi^2(\varepsilon)}dv + \\
&+(x^2-\varphi^2(\varepsilon))\int_{x-\varepsilon}^{x+\varepsilon} \tfrac{v-x}{(v-x)^2+\varphi^2(\varepsilon)}dv - \\
&-\varphi^2(\varepsilon)x\int_{x-\varepsilon}^{x+\varepsilon} \tfrac{1}{(v-x)^2+\varphi^2(\varepsilon)}dv = \\
&= 2x\varepsilon - 2x\varphi(\varepsilon)\left(\arctan\tfrac{\varepsilon}{\varphi(\varepsilon)} - \arctan\tfrac{-\varepsilon}{\varphi(\varepsilon)}\right) + \\
&+(x^2-\varphi^2(\varepsilon))\int_{x-\varepsilon}^{x+\varepsilon} \tfrac{v-x}{(v-x)^2+\varphi^2(\varepsilon)}dv = \\
&= 2x\varepsilon - 4x\varphi(\varepsilon)\arctan\tfrac{\varepsilon}{\varphi(\varepsilon)}.
\end{aligned} \tag{31}
$$

In the equalities (31) we have used (30).

For the third integral in the right side of the equality (29) after straightforward calculations, using (30) and calculations in (31) we obtain that

$$
\begin{aligned}
&\int_{x-\varepsilon}^{x+\varepsilon} \tfrac{(1+v^2)\varphi(\varepsilon)}{(v-x)^2+\varphi^2(\varepsilon)}dv = \varphi(\varepsilon)\int_{x-\varepsilon}^{x+\varepsilon} \tfrac{1}{(v-x)^2+\varphi^2(\varepsilon)}dv + \\
&+\varphi(\varepsilon)\int_{x-\varepsilon}^{x+\varepsilon} \tfrac{v^2}{(v-x)^2+\varphi^2(\varepsilon)}dv = \\
&= 2\arctan\tfrac{\varepsilon}{\varphi(\varepsilon)} + \varphi(\varepsilon)\int_{x-\varepsilon}^{x+\varepsilon} dv - \varphi^3(\varepsilon)\int_{x-\varepsilon}^{x+\varepsilon} \tfrac{1}{(v-x)^2+\varphi^2(\varepsilon)}dv + \\
&+2x\int_{x-\varepsilon}^{x+\varepsilon} \tfrac{v-x}{(v-x)^2+\varphi^2(\varepsilon)}dv + x^2\varphi(\varepsilon)\int_{x-\varepsilon}^{x+\varepsilon} \tfrac{1}{(v-x)^2+\varphi^2(\varepsilon)}dv = \\
&= 2\arctan\tfrac{\varepsilon}{\varphi(\varepsilon)} + 2\varepsilon\varphi(\varepsilon) + 2(x^2-\varphi^2(\varepsilon))\arctan\tfrac{\varepsilon}{\varphi(\varepsilon)}.
\end{aligned} \tag{32}
$$

Now from (29), (30), (31), (32) it follows that

$$
\begin{aligned}
&-i\int_{x-\varepsilon}^{x+\varepsilon} \tfrac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}dv = -i(2x\varepsilon - 4x\varphi(\varepsilon)\arctan\tfrac{\varepsilon}{\varphi(\varepsilon)}) - \\
&-2(1+x^2-\varphi^2(\varepsilon))\arctan\tfrac{\varepsilon}{\varphi(\varepsilon)} - 2\varepsilon\varphi(\varepsilon).
\end{aligned} \tag{33}
$$

But from the form of $\varphi(\varepsilon)$ it follows that

$$\lim_{\varepsilon\to 0}\frac{\varphi(\varepsilon)}{\varepsilon}=0 \tag{34}$$

which together with the equality (33) implies that

$$\lim_{\varepsilon\to 0}\left(-i\int\limits_{x-\varepsilon}^{x+\varepsilon}\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}dv\right)=-\pi(1+x^2). \tag{35}$$

Consequently from (35) it follows that

$$\left\|e^{-iT(x)\int\limits_{x-\varepsilon}^{x+\varepsilon}\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}dv}-e^{-\pi(1+x^2)T(x)}\right\|\longrightarrow 0 \qquad as\ \varepsilon\to 0. \tag{36}$$

Let us denote now

$$\mu=\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}.$$

After direct calculations we obtain that

$$\frac{\mu T(v)-(\mu T(v))^*}{i}=\frac{\mu-\overline{\mu}}{i}T(v)=-2\frac{\varphi(\varepsilon)(1+v^2)}{(v-x)^2+\varphi^2(\varepsilon)}T(v), \tag{37}$$

which implies that $\frac{\mu T(v)-(\mu T(v))^*}{i}$ is nonpositive matrix function. On the other hand

$$\frac{1}{i}\left(\frac{1+xv}{v-x}T(v)-\left(\frac{1+xv}{v-x}T(v)\right)^*\right)=0. \tag{38}$$

Now from (37), (38) and Lemma 3 it follows that

$$\begin{aligned}&\left\|\overset{\rightarrow}{\int\limits_{a}^{x-\varepsilon}}e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}T(v)dv}-\overset{\rightarrow}{\int\limits_{a}^{x-\varepsilon}}e^{-i\frac{1+vx}{v-x}T(v)dv}\right\|\le\\ &\le\int\limits_{a}^{x-\varepsilon}\left|\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}-\frac{1+vx}{v-x}\right|.||T(v)||dv.\end{aligned} \tag{39}$$

Analogously we obtain also the inequality

$$\left\| \overset{\rightarrow}{\int\limits_{x+\varepsilon}^{b}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}} T(v)dv - \overset{\rightarrow}{\int\limits_{x+\varepsilon}^{b}} e^{-i\frac{1+vx}{v-x}} T(v)dv \right\| \leq \tag{40}$$
$$\leq \int\limits_{x+\varepsilon}^{b} \left| \frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)} - \frac{1+vx}{v-x} \right| .||T(v)||dv.$$

But direct calculations show that

$$\left| \frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)} - \frac{1+vx}{v-x} \right| = \frac{\varphi(\varepsilon)(1+v^2)}{|v-x+i\varphi(\varepsilon)|.|v-x|} \leq \tag{41}$$
$$\leq \frac{\varphi(\varepsilon)(1+v^2)}{(v-x)^2}.$$

Further using the relations (39), (40), (41) for the first two summands of the right hand side of (28) we have

$$\left\| \overset{\rightarrow}{\int\limits_{a}^{x-\varepsilon}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}} T(v)dv - \overset{\rightarrow}{\int\limits_{a}^{x-\varepsilon}} e^{-i\frac{1+vx}{v-x}} T(v)dv \right\| +$$
$$+ \left\| \overset{\rightarrow}{\int\limits_{x+\varepsilon}^{b}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}} T(v)dv - \overset{\rightarrow}{\int\limits_{x+\varepsilon}^{b}} e^{-i\frac{1+vx}{v-x}} T(v)dv \right\| \leq$$
$$\leq \int\limits_{a}^{x-\varepsilon} \frac{\varphi(\varepsilon)(1+v^2)}{(v-x)^2} ||T(v)||dv + \int\limits_{x+\varepsilon}^{b} \frac{\varphi(\varepsilon)(1+v^2)}{(v-x)^2} ||T(v)||dv \leq \tag{42}$$
$$\leq 2\varphi(\varepsilon) \int\limits_{a}^{x-\varepsilon} \frac{1+v^2}{(v-x)^2+\varepsilon^2} ||T(v)||dv +$$
$$+2\varphi(\varepsilon) \int\limits_{x+\varepsilon}^{b} \frac{1+v^2}{(v-x)^2+\varepsilon^2} ||T(v)||dv \leq$$
$$\leq 2\varphi(\varepsilon) \int\limits_{a}^{b} \frac{1+v^2}{(v-x)^2+\varepsilon^2} ||T(v)||dv.$$

But for almost all x there exists the limit ([5])

$$\lim_{\varepsilon \to 0} \left(\varepsilon \int\limits_{a}^{b} \frac{1+v^2}{(v-x)^2+\varepsilon^2} ||T(v)||dv \right)$$

which together with the limit (34) imply that

$$2\varphi(\varepsilon)\int_a^b \frac{1+v^2}{(v-x)^2+\varepsilon^2}||T(v)||dv \longrightarrow 0 \quad as \ \varepsilon \to 0. \tag{43}$$

Consequently,

$$\left\| \overset{\rightarrow}{\int_a^{x-\varepsilon}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}}T(v)dv - \int_a^{\overset{\rightarrow}{x-\varepsilon}} e^{-i\frac{1+vx}{v-x}}T(v)dv \right\| + \\ + \left\| \overset{\rightarrow}{\int_{x+\varepsilon}^{b}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}}T(v)dv - \int_{x+\varepsilon}^{\vec{b}} e^{-i\frac{1+vx}{v-x}}T(v)dv \right\| \longrightarrow 0 \tag{44}$$

as $\varepsilon \to 0$, where we have used the inequality (42) and the relation (43).

But from the inequality (37) we have also

$$\frac{\mu T(v)-(\mu T(v))^*}{i} = -2\frac{\varphi(\varepsilon)(1+v^2)}{(v-x)^2+\varphi^2(\varepsilon)}T(v) \leq 0, \tag{45}$$

$$\frac{\mu T(x)-(\mu T(x))^*}{i} = -2\frac{\varphi(\varepsilon)(1+v^2)}{(v-x)^2+\varphi^2(\varepsilon)}T(x) \leq 0. \tag{46}$$

From the relations (45), (46) and Lemma 3 for the third summand on the right hand side of (28) it follows that

$$\left\| \overset{\rightarrow}{\int_{x-\varepsilon}^{x+\varepsilon}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}}T(v)dv - e^{-iT(x)\int_{x-\varepsilon}^{x+\varepsilon}\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}dv} \right\| \leq \\ \leq \int_{x-\varepsilon}^{x+\varepsilon} \left| \frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)} \right| .||T(v)-T(x)||dv. \tag{47}$$

But

$$\left|\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}\right| \leq \frac{1+(M+\varphi(\varepsilon))M}{\varphi(\varepsilon)}$$

where $M = \max\{|a|, |b|\}$. Consequently, for the right hand side of the inequality (47) we have

$$\begin{aligned}
&\int\limits_{x-\varepsilon}^{x+\varepsilon} \left| \frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)} \right| .||T(v)-T(x)||dv. \leq \\
&\leq \frac{1+(M+\varphi(\varepsilon))M}{\varphi(\varepsilon)} \int\limits_{x-\varepsilon}^{x+\varepsilon} ||T(v)-T(x)||dv = \\
&= (1+(M+\varphi(\varepsilon))M) \frac{\int\limits_{x-\varepsilon}^{x+\varepsilon} ||T(v)-T(x)||dv}{\varepsilon^2 + \sqrt{\varepsilon \int\limits_{x-\varepsilon}^{x+\varepsilon} ||T(v)-T(x)||dv}} \leq \\
&\leq (1+(M+\varphi(\varepsilon))M) \frac{1}{\sqrt{\varepsilon}} \sqrt{\int\limits_{x-\varepsilon}^{x+\varepsilon} ||T(v)-T(x)||dv} \longrightarrow 0
\end{aligned} \tag{48}$$

as $\varepsilon \to 0$, according to the choice of x. Consequently, from (48) and (47) it follows that

$$\left\| \overrightarrow{\int\limits_{x-\varepsilon}^{x+\varepsilon}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}T(v)dv} - e^{-iT(x)\int\limits_{x-\varepsilon}^{x+\varepsilon}\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}dv} \right\| \longrightarrow 0 \tag{49}$$

as $\varepsilon \to 0$.

Now from (28) using the relations (36), (42), (43), (49) it follows that

$$\begin{aligned}
&\left\| \overrightarrow{\int\limits_{a}^{b}} e^{-i\frac{1+(x-i\varphi(\varepsilon))v}{v-x+i\varphi(\varepsilon)}T(v)dv} - \right. \\
&\left. - \overrightarrow{\int\limits_{a}^{x-\varepsilon}} e^{-i\frac{1+vx}{v-x}T(v)dv} e^{\pm\pi(1+x^2)T(x)} \overrightarrow{\int\limits_{x+\varepsilon}^{b}} e^{-i\frac{1+vx}{v-x}T(v)dv} \right\| \longrightarrow 0
\end{aligned} \tag{50}$$

as $\varepsilon \to 0$.

Hence the existence of the limit (23) together wit the relation (50) implies that

(51)
$$
\begin{aligned}
& s-\lim_{\delta\to 0}\overset{\overset{b}{\to}}{\underset{a}{\int}} e^{-i\frac{1+(x-i\delta)v}{v-x+i\delta}T(v)dv} = \\
& = s-\lim_{\varepsilon\to 0}\overset{\overset{x-\varepsilon}{\to}}{\underset{a}{\int}} e^{-i\frac{1+vx}{v-x}T(v)dv} e^{-\pi(1+x^2)T(x)} \overset{\overset{b}{\to}}{\underset{x+\varepsilon}{\int}} e^{-i\frac{1+vx}{v-x}T(v)dv}.
\end{aligned}
$$

Analogously it can be obtain the limit

$$
s-\lim_{\delta\to 0}\overset{\overset{b}{\to}}{\underset{a}{\int}} e^{-i\frac{1+(x+i\delta)v}{v-x-i\delta}T(v)dv}.
$$

The theorem is proved. □

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