RIEMANNIAN SPACES OF COMPOSITIONS OF THREE BASIC MANIFOLDS

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ABSTRACT: *We consider Riemannian spaces of compositions of three basic manifolds such that the tangent planes of each manifold are translated parallelly along any line in the other two manifolds. We obtain invariant characteristics of these compositions and explore Riemannian spaces containing such compositions.*

KEYWORDS: *Riemannian space, basic manifold, composition, affinor, tangent planes translation.*

1 Introduction

Symmetric spaces with affine connections, Weyl spaces and Riemannian spaces with additional structures are studied in [1, 4, 5, 6, 7, 9, 10, 11, 14]. Additional structures on such spaces are defined by tensors of type (1,1). Using n independent vector fields, in [16, 17] there is constructed an apparatus for the study of spaces with symmetric connections containing special compositions or special nets. This apparatus is applied to triples of compositions in [2] and to almost para-contact and almost para-complex structures in [12]. Four-dimensional spaces with a symmetric affine connection and additional structures are studied in [3]. In [13], such structures are studied by help of tensors.

Using the apparatus above, we explore Riemannian spaces of compositions of three basic manifolds. In particular, we analyze compositions such that the tangent planes of each basic manifold are translated parallelly along any line in the other two basic manifolds. In a special coordinate net, we define Riemannian spaces consisting of such compositions. We provide an example of such compositions in a three-dimensional Riemannian space.

2 Preliminaries

Let V_N be a Riemannian space with metric tensor $g_{\alpha\beta}$. Denote by $\Gamma^{\nu}_{\alpha\beta}$ and ∇ the coefficients of the Levi-Civita connection and the covariant derivative defined by this connection. Further in this work the following notation indices will be used

$$\begin{aligned}
&\alpha, \beta, \gamma, \sigma, \tau, \nu = 1, 2, \dots, N,\\
&i, j, k, l, s = 1, 2, \dots, m;\ m < N\\
&\bar{i}, \bar{j}, \bar{k}, \bar{l}, \bar{s} = m+1, m+2, \dots, N,\\
&p, q, r, t = m+1, m+2, \dots, m+n;\ m+n < N, \qquad (1)\\
&\bar{p}, \bar{q}, \bar{r}, \bar{t} = 1, 2, ..., m; m+n+1, m+n+2, \dots, N,\\
&a, b, c, d = m+n+1, m+n+2, \dots, N,\\
&\bar{a}, \bar{b}, \bar{c}, \bar{d} = 1, 2, \dots, m+n.
\end{aligned}$$

Let $\underset{\alpha}{v}^{\beta}$ be independent vector fields, and $\{\underset{\alpha}{v}\}$ be the net defined by $\underset{\alpha}{v}^{\beta}$. The reciprocal covector fields $\overset{\alpha}{v}_{\beta}$ to $\underset{\alpha}{v}^{\beta}$ are given by

$$\underset{\alpha}{v}^{\beta}\overset{\nu}{v}_{\beta} = \delta^{\nu}_{\alpha} \quad \Leftrightarrow \quad \underset{\beta}{v}^{\nu}\overset{\beta}{v}_{\alpha} = \delta^{\nu}_{\alpha} \qquad (2)$$

where δ^{ν}_{α} is the identity affinor. If $\{\underset{\alpha}{v}\}$ is the coordinate net, then $g_{\alpha\beta}\underset{\nu}{v}^{\alpha}\underset{\nu}{v}^{\beta} = 1$ and (2) imply that the vectors $\underset{\alpha}{v}^{\beta}$ and their reciprocal covectors $\overset{\alpha}{v}_{\beta}$ are given by:

$$\begin{aligned}
&\underset{1}{v}^{\alpha}\left(\frac{1}{\sqrt{g_{11}}}, 0, \dots, 0\right), \underset{2}{v}^{\alpha}\left(0, \frac{1}{\sqrt{g_{22}}}, \dots, 0\right), \dots, \underset{N}{v}^{\alpha}\left(0, 0, \dots, \frac{1}{\sqrt{g_{NN}}}\right),\\
&\overset{1}{v}_{\alpha}\left(\sqrt{g_{11}}, 0, \dots, 0\right), \overset{2}{v}_{\alpha}\left(0, \sqrt{g_{22}}, 0, \dots, 0\right), \dots, \overset{N}{v}_{\alpha}\left(0, 0, \dots, \sqrt{g_{NN}}\right).
\end{aligned} \qquad (3)$$

In addition to the arbitrary coordinates x^{α} $(\alpha = 1, \dots, N)$, in V_N we introduce the coordinates $\overset{\alpha}{u}$ with respect to the net $\{\underset{\alpha}{v}\}$. For a given vector field v^{α} we have $v^{\alpha}(\overset{1}{u}, \overset{2}{u}, \dots, \overset{N}{u})$.

The following derivative equations are known to be valid [15, 16]

$$\nabla_\sigma \underset{\alpha}{v}^\beta = \overset{\nu}{\underset{\alpha}{T}}{}_\sigma \underset{\nu}{v}^\beta, \quad \nabla_\sigma \overset{\alpha}{v}_\beta = -\overset{\alpha}{\underset{\nu}{T}}{}_\sigma \overset{\nu}{v}_\beta. \tag{4}$$

From (2), (3), (4) and $\nabla_\sigma \underset{\alpha}{v}^\beta = \partial_\sigma \underset{\alpha}{v}^\beta + \Gamma^\beta_{\sigma\nu} \underset{\alpha}{v}^\nu$ in the parameters of the coordinate net $\{\underset{\alpha}{v}\}$ the following hold

$$\overset{\nu}{\underset{\alpha}{T}}{}_\sigma = \Gamma^\nu_{\sigma\alpha} \frac{\sqrt{g_{\nu\nu}}}{\sqrt{g_{\alpha\alpha}}}, \quad \alpha \neq \nu, \tag{5}$$

$$\overset{\alpha}{\underset{\alpha}{T}}{}_\sigma = \Gamma^\alpha_{\sigma\alpha} - \frac{1}{2}\partial_\sigma \ln(g_{11}, g_{22}, \ldots, g_{NN}). \tag{6}$$

If g is the determinant of $(g_{\alpha\beta})$ then

$$\Gamma^\alpha_{\sigma\alpha} = \partial_\sigma \ln\sqrt{|g|} = \frac{\partial \ln\sqrt{|g|}}{\partial \overset{\sigma}{u}}. \tag{7}$$

From (6) and (7) we find

$$\overset{\alpha}{\underset{\alpha}{T}}{}_\sigma = \partial_\sigma \ln \frac{\ln\sqrt{|g|}}{\sqrt{g_{11}g_{22}\cdots g_{NN}}}. \tag{8}$$

Therefore $\overset{\alpha}{\underset{\alpha}{T}}{}_\sigma = \mathrm{grad}$.

Further, consider the affinors [16, 17]

$$\begin{aligned} a^\beta_\alpha &= \underset{i}{v}^\beta \overset{i}{v}_\alpha - \underset{\bar{i}}{v}^\beta \overset{\bar{i}}{v}_\alpha, \\ b^\beta_\alpha &= \underset{p}{v}^\beta \overset{p}{v}_\alpha - \underset{\bar{p}}{v}^\beta \overset{\bar{p}}{v}_\alpha, \\ c^\beta_\alpha &= \underset{a}{v}^\beta \overset{a}{v}_\alpha - \underset{\bar{a}}{v}^\beta \overset{\bar{a}}{v}_\alpha. \end{aligned} \tag{9}$$

From (2) and (9) we have $a^\beta_\alpha a^\nu_\beta = \delta^\nu_\alpha$, $b^\beta_\alpha b^\nu_\beta = \delta^\nu_\alpha$ and $c^\beta_\alpha c^\nu_\beta = \delta^\nu_\alpha$. From (6) and the last equations it follows that the affinors a^β_α, b^β_α and c^β_α

define the compositions $X_m \times X_{N-m}$, $X_n \times X_{N-n}$ and $X_{N-m-n} \times X_{n+m}$, respectively.

Denote by $P(X_m)$, $P(X_{N-m})$, $P(X_n)$, $P(X_{N-n})$, $P(X_{N-m-n})$ and $P(X_{n+m})$ [7] the positions (tangent spaces) to the basic manifolds X_m, X_{N-m}, X_n, X_{N-n}, X_{N-m-n} and X_{n+m} respectively. Denote by $\overset{1}{a}{}_\alpha^\beta$ and $\overset{2}{a}{}_\alpha^\beta$, $\overset{1}{b}{}_\alpha^\beta$ and $\overset{2}{b}{}_\alpha^\beta$, $\overset{1}{c}{}_\alpha^\beta$ and $\overset{2}{c}{}_\alpha^\beta$ the projecting affinors of the compositions $X_m \times X_{N-m}$, $X_n \times X_{N-n}$ and $X_{N-m-n} \times X_{n+m}$, respectively [16, 17]. According to [16, 17], we have

$$\overset{1}{a}{}_\alpha^\beta = \underset{i}{v}{}^\beta \overset{i}{v}_\alpha,\ \overset{2}{a}{}_\alpha^\beta = \underset{\bar{i}}{v}{}^\beta \overset{\bar{i}}{v}_\alpha,\ \overset{1}{b}{}_\alpha^\beta = \underset{p}{v}{}^\beta \overset{p}{v}_\alpha,\ \overset{2}{b}{}_\alpha^\beta = \underset{\bar{p}}{v}{}^\beta \overset{\bar{p}}{v}_\alpha,\ \overset{1}{c}{}_\alpha^\beta = \underset{a}{v}{}^\beta \overset{a}{v}_\alpha,\ \overset{2}{c}{}_\alpha^\beta = \underset{\bar{a}}{v}{}^\beta \overset{\bar{a}}{v}_\alpha. \tag{10}$$

Consider the composition $X_m \times X_n \times X_{N-m-n}$ of three basic manifolds X_m, X_n and X_{N-m-n} i.e. their topological product. The tangent spaces of these basic manifolds are $P(X_m), P(X_n)$ and $P(X_{N-m-n})$, and $\overset{1}{a}{}_\alpha^\beta$, $\overset{1}{b}{}_\alpha^\beta$ and $\overset{1}{c}{}_a^\beta$ are the projecting affinors of these manifolds. According to [16, 17], we have

$$\begin{aligned} &\overset{1}{a}{}_\alpha^\beta + \overset{1}{b}{}_\alpha^\beta + \overset{1}{c}{}_\alpha^\beta = \delta_\alpha^\beta, \quad \overset{1}{a}{}_\alpha^\beta \overset{1}{a}{}_\beta^\nu = \overset{1}{a}{}_\alpha^\nu, \quad \overset{1}{b}{}_\alpha^\beta \overset{1}{b}{}_\beta^\nu = \overset{1}{b}{}_\alpha^\nu, \quad \overset{1}{c}{}_\alpha^\beta \overset{1}{c}{}_\beta^\nu = \overset{1}{c}{}_\alpha^\nu, \\ &\overset{1}{a}{}_\alpha^\beta \overset{1}{b}{}_\beta^\nu = \overset{1}{b}{}_\alpha^\beta \overset{1}{a}{}_\beta^\nu = \overset{1}{a}{}_\alpha^\beta \overset{1}{c}{}_\beta^\nu = \overset{1}{c}{}_\alpha^\beta \overset{1}{a}{}_\beta^\nu = \overset{1}{b}{}_\alpha^\beta \overset{1}{c}{}_\beta^\nu = \overset{1}{c}{}_\alpha^\beta \overset{1}{b}{}_\beta^\nu = 0. \end{aligned} \tag{11}$$

If v^α is a random vector, then $v^\alpha = \overset{1}{a}{}_\beta^\alpha v^\beta + \overset{1}{b}{}_\beta^\alpha v^\beta + \overset{1}{c}{}_\beta^\alpha v^\beta = \overset{1}{V}{}^\alpha + \overset{2}{V}{}^\alpha + \overset{3}{V}{}^\alpha$, where $\overset{1}{V}{}^\alpha = \overset{1}{a}{}_\beta^\alpha v^\beta \in P(X_m)$, $\overset{2}{V}{}^\alpha = \overset{1}{b}{}_\beta^\alpha v^\beta \in P(X_n)$, $\overset{3}{V}{}^\alpha = \overset{1}{c}{}_\beta^\alpha v^\beta \in P(X_{N-m-n})$. Obviously $\underset{i}{v}{}^\alpha \in P(X_m)$, $\underset{p}{v}{}^\alpha \in P(X_n)$, and $\underset{a}{v}{}^\alpha \in P(X_{N-m-n})$.

If $R_{\alpha\beta\gamma}{}^\nu$ and $R_{\alpha\beta}$ are the curvature tensor and the Ricci tensor respectively, then [8]

$$\begin{aligned} R_{\alpha\beta\gamma}{}^\nu &= \partial_\alpha \Gamma_{\beta\gamma}^\nu - \partial_\beta \Gamma_{\alpha\gamma}^\nu + \Gamma_{\alpha\tau}^\nu \Gamma_{\beta\gamma}^\tau - \Gamma_{\beta\tau}^\nu \Gamma_{\alpha\gamma}^\tau, \\ R_{\alpha\beta} &= \partial_\nu \Gamma_{\alpha\beta}^\nu - \partial_\alpha \Gamma_{\nu\beta}^\nu + \Gamma_{\nu\tau}^\nu \Gamma_{\alpha\beta}^\tau - \Gamma_{\nu\alpha}^\tau \Gamma_{\tau\beta}^\nu. \end{aligned} \tag{12}$$

3 Riemann Spaces of Compositions $X_m \times X_n \times X_{N-m-n}$

Consider the affinor

$$B_\alpha^\beta = \underset{i}{v}^{\beta}\overset{i}{v}_{\alpha} - \underset{p}{v}^{\beta}\overset{p}{v}_{\alpha}. \tag{13}$$

From (2) and (13) we have $B_\alpha^\beta B_\beta^\nu = \underset{i}{v}^{\nu}\overset{i}{v}_{\alpha} + \underset{p}{v}^{\nu}\overset{p}{v}_{\alpha} = \delta_\alpha^\nu - \underset{a}{v}^{\nu}\overset{a}{v}_{\alpha}$. In the parameters of the coordinate net, the matrix of the affinor B_α^β is given by

$$(\mathbf{B}_\alpha^\beta) = \begin{pmatrix} \delta_i^j & 0 & 0 \\ 0 & -\delta_p^q & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

Definition 1 *The composition $X_m \times X_n \times X_{N-m-n}$ is a Z-composition provided*

$$\nabla_{[\sigma} B_{\alpha]}^\beta = 0. \tag{15}$$

Theorem 1 *The Riemannian space V_N with metric tensor $g_{\alpha\beta}$ is a space of Z-compositions $X_m \times X_n \times X_{N-m-n}$ if and only if the coefficients of derivative equations (4) and the independent vector fields $\underset{\nu}{v}^{\alpha}$ satisfy*

$$\begin{gathered}
\underset{p}{\overset{i}{T}}{}_{\sigma}\underset{q}{v}^{\sigma} - \underset{q}{\overset{i}{T}}{}_{\sigma}\underset{p}{v}^{\sigma} = 0,\ \underset{b}{\overset{i}{T}}{}_{\sigma}\underset{a}{v}^{\sigma} - \underset{a}{\overset{i}{T}}{}_{\sigma}\underset{b}{v}^{\sigma} = 0,\ \underset{s}{\overset{p}{T}}{}_{\sigma}\underset{k}{v}^{\sigma} - \underset{k}{\overset{p}{T}}{}_{\sigma}\underset{s}{v}^{\sigma} = 0, \\
\underset{a}{\overset{p}{T}}{}_{\sigma}\underset{b}{v}^{\sigma} - \underset{b}{\overset{p}{T}}{}_{\sigma}\underset{a}{v}^{\sigma} = 0,\ \underset{s}{\overset{a}{T}}{}_{\sigma}\underset{k}{v}^{\sigma} - \underset{k}{\overset{a}{T}}{}_{\sigma}\underset{s}{v}^{\sigma} = 0,\ \underset{p}{\overset{a}{T}}{}_{\sigma}\underset{q}{v}^{\sigma} - \underset{q}{\overset{a}{T}}{}_{\sigma}\underset{p}{v}^{\sigma} = 0, \\
\underset{s}{\overset{a}{T}}{}_{\sigma}\underset{p}{v}^{\sigma} + \underset{p}{\overset{a}{T}}{}_{\sigma}\underset{s}{v}^{\sigma} = 0,\ 2\ \underset{s}{\overset{p}{T}}{}_{\sigma}\underset{a}{v}^{\sigma} - \underset{a}{\overset{p}{T}}{}_{\sigma}\underset{s}{v}^{\sigma} = 0,\ 2\ \underset{p}{\overset{i}{T}}{}_{\sigma}\underset{a}{v}^{\sigma} - \underset{a}{\overset{i}{T}}{}_{\sigma}\underset{p}{v}^{\sigma} = 0, \\
\underset{p}{\overset{i}{T}}{}_{\sigma}\underset{s}{v}^{\sigma} = \underset{s}{\overset{p}{T}}{}_{\sigma}\underset{q}{v}^{\sigma} = \underset{a}{\overset{i}{T}}{}_{\sigma}\underset{s}{v}^{\sigma} = \underset{s}{\overset{a}{T}}{}_{\sigma}\underset{b}{v}^{\sigma} = \underset{a}{\overset{p}{T}}{}_{\sigma}\underset{q}{v}^{\sigma} = \underset{p}{\overset{b}{T}}{}_{\sigma}\underset{a}{v}^{\sigma} = 0.
\end{gathered} \tag{16}$$

Proof Using (4), equation (15) can be written as

$$\begin{aligned} 2\nabla_{[\sigma}B^{\beta}_{\alpha]} = {} & \overset{\nu}{\underset{i}{T}}_{\sigma}\underset{\nu}{v}^{\beta}\overset{i}{v}_{\alpha} - \overset{\nu}{\underset{i}{T}}_{\alpha}\underset{\nu}{v}^{\beta}\overset{i}{v}_{\sigma} - \overset{i}{\underset{\nu}{T}}_{\sigma}\underset{i}{v}^{\beta}\overset{\nu}{v}_{\alpha} + \overset{i}{\underset{\nu}{T}}_{\alpha}\underset{i}{v}^{\beta}\overset{\nu}{v}_{\sigma} \\ & - \overset{\nu}{\underset{p}{T}}_{\sigma}\underset{\nu}{v}^{\beta}\overset{p}{v}_{\alpha} + \overset{\nu}{\underset{p}{T}}_{\alpha}\underset{\nu}{v}^{\beta}\overset{p}{v}_{\sigma} + \overset{p}{\underset{\nu}{T}}_{\sigma}\underset{p}{v}^{\beta}\overset{\nu}{v}_{\alpha} - \overset{p}{\underset{\nu}{T}}_{\alpha}\underset{p}{v}^{\beta}\overset{\nu}{v}_{\sigma} = 0. \end{aligned} \tag{17}$$

According to (2), after contracting the last equation with $\underset{s}{v}^{\alpha}$, $\underset{q}{v}^{\alpha}$ and $\underset{a}{v}^{\alpha}$, we obtain the following equations equivalent to (17)

$$\begin{aligned} & \overset{\nu}{\underset{s}{T}}_{\sigma}\underset{\nu}{v}^{\beta} - \overset{\nu}{\underset{i}{T}}_{\alpha}\underset{s}{v}^{\alpha}\underset{\nu}{v}^{\beta}\overset{i}{v}_{\sigma} - \overset{i}{\underset{s}{T}}_{\sigma}\underset{i}{v}^{\beta} + \overset{i}{\underset{\nu}{T}}_{\alpha}\underset{s}{v}^{\alpha}\underset{i}{v}^{\beta}\overset{\nu}{v}_{\sigma} + \overset{\nu}{\underset{p}{T}}_{\alpha}\underset{s}{v}^{\alpha}\underset{\nu}{v}^{\beta}\overset{p}{v}_{\sigma} + \overset{p}{\underset{s}{T}}_{\sigma}\underset{p}{v}^{\beta} - \\ & - \overset{p}{\underset{\nu}{T}}_{\alpha}\underset{s}{v}^{\alpha}\underset{p}{v}^{\beta}\overset{\nu}{v}_{\sigma} = 0, \\ & - \overset{\nu}{\underset{i}{T}}_{\alpha}\underset{q}{v}^{\alpha}\underset{\nu}{v}^{\beta}\overset{i}{v}_{\sigma} - \overset{i}{\underset{q}{T}}_{\sigma}\underset{i}{v}^{\beta} + \overset{i}{\underset{\nu}{T}}_{\alpha}\underset{q}{v}^{\alpha}\underset{i}{v}^{\beta}\overset{\nu}{v}_{\sigma} - \overset{\nu}{\underset{q}{T}}_{\sigma}\underset{\nu}{v}^{\beta} + \overset{\nu}{\underset{p}{T}}_{\alpha}\underset{q}{v}^{\alpha}\underset{\nu}{v}^{\beta}\overset{p}{v}_{\sigma} + \overset{p}{\underset{q}{T}}_{\sigma}\underset{p}{v}^{\beta} - \\ & - \overset{p}{\underset{\nu}{T}}_{\alpha}\underset{q}{v}^{\alpha}\underset{p}{v}^{\beta}\overset{\nu}{v}_{\sigma} = 0, \\ & - \overset{\nu}{\underset{i}{T}}_{\alpha}\underset{a}{v}^{\alpha}\underset{\nu}{v}^{\beta}\overset{i}{v}_{\sigma} - \overset{i}{\underset{a}{T}}_{\sigma}\underset{i}{v}^{\beta} + \overset{i}{\underset{\nu}{T}}_{\alpha}\underset{a}{v}^{\alpha}\underset{i}{v}^{\beta}\overset{\nu}{v}_{\sigma} + \overset{\nu}{\underset{p}{T}}_{\alpha}\underset{a}{v}^{\alpha}\underset{\nu}{v}^{\beta}\overset{p}{v}_{\sigma} + \overset{p}{\underset{a}{T}}_{\sigma}\underset{p}{v}^{\beta} - \\ & - \overset{p}{\underset{\nu}{T}}_{\alpha}\underset{a}{v}^{\alpha}\underset{p}{v}^{\beta}\overset{\nu}{v}_{\sigma} = 0. \end{aligned} \tag{18}$$

Contract the first equation of (18) with $\underset{\alpha}{v}^{\sigma}$, $\underset{r}{v}^{\sigma}$ and $\underset{b}{v}^{\sigma}$ to obtain

$$\begin{aligned} & \overset{\nu}{\underset{s}{T}}_{\sigma}\underset{k}{v}^{\sigma}\underset{\nu}{v}^{\beta} - \overset{\nu}{\underset{k}{T}}_{\alpha}\underset{s}{v}^{\alpha}\underset{\nu}{v}^{\beta} - \overset{i}{\underset{s}{T}}_{\sigma}\underset{k}{v}^{\sigma}\underset{i}{v}^{\beta} + \overset{i}{\underset{k}{T}}_{\alpha}\underset{s}{v}^{\alpha}\underset{i}{v}^{\beta} + \overset{p}{\underset{s}{T}}_{\sigma}\underset{k}{v}^{\sigma}\underset{p}{v}^{\beta} - \overset{p}{\underset{k}{T}}_{\alpha}\underset{s}{v}^{\alpha}\underset{p}{v}^{\beta} = 0, \\ & \overset{\nu}{\underset{s}{T}}_{\sigma}\underset{r}{v}^{\sigma}\underset{\nu}{v}^{\beta} - \overset{i}{\underset{s}{T}}_{\sigma}\underset{r}{v}^{\sigma}\underset{i}{v}^{\beta} + \overset{i}{\underset{r}{T}}_{\alpha}\underset{s}{v}^{\alpha}\underset{i}{v}^{\beta} + \overset{\nu}{\underset{r}{T}}_{\alpha}\underset{s}{v}^{\alpha}\underset{\nu}{v}^{\beta} + \overset{p}{\underset{s}{T}}_{\sigma}\underset{r}{v}^{\sigma}\underset{p}{v}^{\beta} - \overset{p}{\underset{r}{T}}_{\alpha}\underset{s}{v}^{\alpha}\underset{p}{v}^{\beta} = 0, \\ & \overset{\nu}{\underset{s}{T}}_{\sigma}\underset{b}{v}^{\sigma}\underset{\nu}{v}^{\beta} - \overset{i}{\underset{s}{T}}_{\sigma}\underset{b}{v}^{\sigma}\underset{i}{v}^{\beta} + \overset{i}{\underset{b}{T}}_{\sigma}\underset{s}{v}^{\sigma}\underset{i}{v}^{\beta} + \overset{p}{\underset{s}{T}}_{\sigma}\underset{b}{v}^{\sigma}\underset{p}{v}^{\beta} - \overset{p}{\underset{b}{T}}_{\sigma}\underset{s}{v}^{\sigma}\underset{p}{v}^{\beta} = 0. \end{aligned} \tag{19}$$

Equations (19) are equivalent to the first equation of (18). Reasoning analogously, the second equation of (18) is equivalent to

$$\begin{aligned}
&-\overset{\nu}{\underset{k}{T}}{}_{\sigma}\underset{q}{v}^{\sigma}\underset{\nu}{v}^{\beta}-\overset{i}{\underset{q}{T}}{}_{\sigma}\underset{k}{v}^{\sigma}\underset{i}{v}^{\beta}+\overset{i}{\underset{k}{T}}{}_{\alpha}\underset{q}{v}^{\alpha}\underset{i}{v}^{\beta}-\overset{\nu}{\underset{a}{T}}{}_{\sigma}\underset{k}{v}^{\sigma}\underset{\nu}{v}^{\beta}+\overset{p}{\underset{q}{T}}{}_{\sigma}\underset{k}{v}^{\sigma}\underset{p}{v}^{\beta}-\overset{p}{\underset{k}{T}}{}_{\alpha}\underset{q}{v}^{\alpha}\underset{p}{v}^{\beta}=0,\\
&-\overset{i}{\underset{q}{T}}{}_{\sigma}\underset{r}{v}^{\sigma}\underset{i}{v}^{\beta}+\overset{i}{\underset{r}{T}}{}_{\alpha}\underset{q}{v}^{\alpha}\underset{i}{v}^{\beta}-\overset{\nu}{\underset{q}{T}}{}_{\sigma}\underset{r}{v}^{\sigma}\underset{\nu}{v}^{\beta}+\overset{\nu}{\underset{r}{T}}{}_{\alpha}\underset{q}{v}^{\alpha}\underset{\nu}{v}^{\beta}+\overset{p}{\underset{q}{T}}{}_{\sigma}\underset{r}{v}^{\sigma}\underset{p}{v}^{\beta}-\overset{p}{\underset{r}{T}}{}_{\alpha}\underset{q}{v}^{\alpha}\underset{p}{v}^{\beta}=0,\\
&-\overset{i}{\underset{q}{T}}{}_{\sigma}\underset{a}{v}^{\sigma}\underset{i}{v}^{\beta}+\overset{i}{\underset{a}{T}}{}_{\alpha}\underset{r}{v}^{\alpha}\underset{i}{v}^{\beta}-\overset{\nu}{\underset{q}{T}}{}_{\sigma}\underset{a}{v}^{\sigma}\underset{\nu}{v}^{\beta}+\overset{p}{\underset{q}{T}}{}_{\sigma}\underset{a}{v}^{\sigma}\underset{p}{v}^{\beta}-\overset{p}{\underset{a}{T}}{}_{\alpha}\underset{r}{v}^{\alpha}\underset{p}{v}^{\beta}=0.
\end{aligned}\tag{20}$$

Similarly, the third equation of (18) is equivalent to

$$\begin{aligned}
&-\overset{\nu}{\underset{k}{T}}{}_{\alpha}\underset{a}{v}^{\alpha}\underset{\nu}{v}^{\beta}-\overset{i}{\underset{a}{T}}{}_{\sigma}\underset{k}{v}^{\sigma}\underset{i}{v}^{\beta}+\overset{i}{\underset{k}{T}}{}_{\alpha}\underset{a}{v}^{\alpha}\underset{i}{v}^{\beta}+\overset{p}{\underset{a}{T}}{}_{\sigma}\underset{k}{v}^{\sigma}\underset{p}{v}^{\beta}-\overset{p}{\underset{k}{T}}{}_{\alpha}\underset{a}{v}^{\alpha}\underset{p}{v}^{\beta}=0,\\
&-\overset{i}{\underset{a}{T}}{}_{\sigma}\underset{r}{v}^{\sigma}\underset{i}{v}^{\beta}+\overset{i}{\underset{r}{T}}{}_{\alpha}\underset{a}{v}^{\alpha}\underset{i}{v}^{\beta}+\overset{\nu}{\underset{r}{T}}{}_{\alpha}\underset{a}{v}^{\alpha}\underset{\nu}{v}^{\beta}+\overset{p}{\underset{a}{T}}{}_{\sigma}\underset{i}{v}^{\sigma}\underset{p}{v}^{\beta}-\overset{p}{\underset{r}{T}}{}_{\alpha}\underset{a}{v}^{\alpha}\underset{p}{v}^{\beta}=0,\\
&-\overset{i}{\underset{a}{T}}{}_{\sigma}\underset{b}{v}^{\sigma}\underset{i}{v}^{\beta}+\overset{i}{\underset{b}{T}}{}_{\alpha}\underset{a}{v}^{\alpha}\underset{i}{v}^{\beta}+\overset{p}{\underset{a}{T}}{}_{\sigma}\underset{b}{v}^{\sigma}\underset{p}{v}^{\beta}-\overset{p}{\underset{b}{T}}{}_{\alpha}\underset{a}{v}^{\alpha}\underset{p}{v}^{\beta}=0.
\end{aligned}\tag{21}$$

Since the vector fields $\underset{\alpha}{v}^{\beta}$ are independent, it follows that equations (19), (20) and (21) are equivalent to (16) which completes the proof.

Corollary 1 *In the parameters of coordinate net $\{\underset{\alpha}{v}\}$, equations (16) imply*

$$\Gamma^i_{pa}=\Gamma^i_{sp}=\Gamma^i_{sa}=0,\ \Gamma^p_{as}=\Gamma^p_{qs}=\Gamma^p_{qa}=0,\ \Gamma^a_{sp}=\Gamma^a_{sb}=\Gamma^a_{bp}=0.\tag{22}$$

Proof According to (3), equations (16) have the form

$$\begin{aligned}
&\frac{1}{\sqrt{g_{kk}}}\overset{p}{\underset{s}{T}}{}_k-\frac{1}{\sqrt{g_{ss}}}\overset{p}{\underset{k}{T}}{}_s=0,\ \frac{1}{\sqrt{g_{kk}}}\overset{a}{\underset{s}{T}}{}_k-\frac{1}{\sqrt{g_{ss}}}\overset{a}{\underset{k}{T}}{}_s=0,\\
&\frac{1}{\sqrt{g_{rr}}}\overset{i}{\underset{q}{T}}{}_r-\frac{1}{\sqrt{g_{qq}}}\overset{i}{\underset{r}{T}}{}_q=0,\ \frac{1}{\sqrt{g_{rr}}}\overset{a}{\underset{q}{T}}{}_r-\frac{1}{\sqrt{g_{qq}}}\overset{a}{\underset{r}{T}}{}_q=0,\\
&\frac{1}{\sqrt{g_{aa}}}\overset{i}{\underset{b}{T}}{}_a-\frac{1}{\sqrt{g_{bb}}}\overset{i}{\underset{a}{T}}{}_b=0,\ \frac{1}{\sqrt{g_{bb}}}\overset{p}{\underset{a}{T}}{}_b-\frac{1}{\sqrt{g_{aa}}}\overset{p}{\underset{b}{T}}{}_a=0,\\
&\frac{2}{\sqrt{g_{bb}}}\overset{p}{\underset{s}{T}}{}_b-\frac{1}{\sqrt{g_{ss}}}\overset{p}{\underset{b}{T}}{}_s=0,\ \frac{2}{\sqrt{g_{aa}}}\overset{i}{\underset{p}{T}}{}_a-\frac{1}{\sqrt{g_{pp}}}\overset{i}{\underset{a}{T}}{}_p=0,\\
&\frac{1}{\sqrt{g_{pp}}}\overset{a}{\underset{k}{T}}{}_p+\frac{1}{\sqrt{g_{kk}}}\overset{a}{\underset{p}{T}}{}_k=0,\ \frac{1}{\sqrt{g_{ss}}}\overset{i}{\underset{r}{T}}{}_s=0,\ \frac{1}{\sqrt{g_{rr}}}\overset{p}{\underset{s}{T}}{}_r=0,\\
&\frac{1}{\sqrt{g_{ss}}}\overset{i}{\underset{a}{T}}{}_s=0,\ \frac{1}{\sqrt{g_{bb}}}\overset{a}{\underset{s}{T}}{}_b=0,\ \frac{1}{\sqrt{g_{qq}}}\overset{p}{\underset{a}{T}}{}_q=0,\ \frac{1}{\sqrt{g_{aa}}}\overset{b}{\underset{p}{T}}{}_a=0.
\end{aligned}\tag{23}$$

Using (5) and the symmetry of $\Gamma^{\nu}_{\alpha\beta}$, equations (23) can be written as (22).

From [7] and (22) it follows that the positions $P(X_m)$, $P(X_n)$ and $P(X_{N-m-n})$ of the Z-composition $X_m \times X_n \times X_{N-m-n}$ are translated parallelly along any line in the manifolds X_n and X_{N_m-n}, X_m and X_{N-m-n}, X_m and X_n. From the fundamental equation $\nabla_\sigma g_{\alpha\beta}=0$ and (22), we obtain the following in the parameters of coordinate net $\{\underset{\alpha}{v}\}$

$$\partial_p g_{ij}=\partial_a g_{ij}=0,\ \partial_i g_{pq}=\partial_a g_{pq}=0,\ \partial_i g_{ab}=\partial_p g_{ab}=0,$$
$$\partial_i g_{pa}=\partial_p g_{ai}=\partial_a g_{ip}=0.$$

Therefore $g_{ij}(\overset{k}{u})$, $g_{pq}(\overset{r}{u})$, $g_{ab}(\overset{c}{u})$, $g_{ip}(\overset{k}{u},\overset{r}{u})$, $g_{ia}(\overset{k}{u},\overset{c}{u})$, $g_{pa}(\overset{r}{u},\overset{c}{u})$. In the parameters of the coordinate net $\{\underset{\alpha}{v}\}$, from (12) and (13) it follows:

$$R_{ip}=-\partial_i\Gamma^q_{qp}-\Gamma^q_{si}\Gamma^s_{qp},\ R_{ia}=-\partial_i\Gamma^b_{ba}-\Gamma^b_{si}\Gamma^s_{ba},$$
$$R_{pa}=-\partial_p\Gamma^b_{ba}-\Gamma^b_{qp}\Gamma^q_{ba},$$
$$R_{ijp}{}^{\alpha}=R_{ija}{}^{\alpha}=0,\ R_{pqs}{}^{\alpha}=R_{pqa}{}^{\alpha}=0,\ R_{abs}{}^{\alpha}=R_{abp}{}^{\alpha}=0,$$
$$R_{ipa}{}^{\alpha}=0.$$

Consider a three-dimensional Riemannian space $V_3(g_{\alpha\beta})$. Let the vector fields $\underset{1}{v}^\alpha$, $\underset{2}{v}^\alpha$ and $\underset{3}{v}^\alpha$ define the tangent spaces of the basic one-dimensional spaces X_1, Y_1 and Z_1 of the Z-composition $X_1 \times Y_1 \times Z_1$. Then $\underset{1}{v}^\alpha \in P(X_1)$, $\underset{2}{v}^\alpha \in P(Y_1)$, $\underset{3}{v}^\alpha \in P(Z_1)$. Equations (22) can be written as

$$\Gamma^1_{23} = \Gamma^1_{12} = \Gamma^1_{13} = 0,\ \Gamma^2_{31} = \Gamma^2_{12} = \Gamma^2_{23} = 0,\ \Gamma^3_{12} = \Gamma^3_{13} = \Gamma^3_{23} = 0.$$

In the parameters of the coordinate net $\{\underset{\alpha}{v}\}$ we have

$$g_{11}(\overset{1}{u}),\ g_{22}(\overset{2}{u}),\ g_{33}(\overset{3}{u}),\ g_{12} = \sqrt{g_{11}}\sqrt{g_{22}}\cos\underset{12}{\omega},$$
$$g_{13} = \sqrt{g_{11}}\sqrt{g_{33}}\cos\underset{13}{\omega},\ g_{23} = \sqrt{g_{22}}\sqrt{g_{33}}\cos\underset{23}{\omega}$$

where $\underset{\alpha\beta}{\omega}$ is the angle between the vector fields $\underset{\alpha}{v}^\nu$ and $\underset{\alpha}{v}^\nu$ and $\underset{\alpha\beta}{\omega}(\overset{\alpha}{u}, \overset{\beta}{u})$. We change the variables as follow: $\sqrt{g_{11}}d\overset{1}{u} = d\overset{1}{U}$, $\sqrt{g_{22}}d\overset{2}{u} = d\overset{2}{U}$, $\sqrt{g_{33}}d\overset{3}{u} = d\overset{3}{U}$. Then the line element of the Riemannian space $V_3(g_{\alpha\beta})$ of the composition $X_1 \times Y_1 \times Z_1$ is given by

$$ds^2 = d\overset{1}{U}{}^2 + d\overset{2}{U}{}^2 + d\overset{3}{U}{}^2 + 2\cos\underset{12}{\omega}d\overset{1}{U}d\overset{2}{U} + 2\cos\underset{13}{\omega}d\overset{1}{U}d\overset{3}{U} + 2\cos\underset{23}{\omega}d\overset{2}{U}d\overset{3}{U}.$$

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