ON FIXED POINTS FOR REICH MAPS IN B-METRIC SPACES

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ABSTRACT: *In this paper we find sufficient conditions for the existence and uniqueness of fixed points for a class of Reich maps in b-metric space. These conditions do not involve the b-metric constant. We establish a priori error estimate for the sequence of successive iterations. The error estimate, which we present is better than that the well-known one for a wide class of Reich maps in metric spaces.*

KEYWORDS: *Fixed point, Hardy-Rogers map, b-Metric space, A priori error estimate*

1. INTRODUCTION

Fixed point theory has got wide applications in different branches of mathematics. Since the work of S. Banach [3] known as the Banach Contraction Principle, many mathematicians have extended and generalized the results in [3]. Some of the classical generalizations of [3] are presented in [15]. The concept of a b-metric space as a generalization of a metric space is introduced in [6] and a contraction mapping theorem is proved there. Since then results about fixed points, variational principles and applications were obtained in b-metric spaces. We will cite just a few recent results in these directions [1, 2, 5, 6, 7, 9, 12, 14, 15, 19].

An extensive study of the problem about fixed points in different distance spaces and references can be found in [11]. Following [11] we recall some definitions and properties for b-metric spaces.

Definition 1.1. Let X be a non-empty set and $s \geq 1$. A functional $\rho : X \times X \to \mathbb{R}$ is called a b-metric if it satisfies the following conditions:

$\rho(x, y) \geq 0$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

$\rho(x,y)=\rho(y,x)$ for all $x,y\in X$;

$\rho(x,y)\le s(\rho(x,z)+\rho(z,y))$ for all $x,y,z\in X$.

The ordered pair (X,ρ) is called a b-metric space (with constant s).

Any metric space is a b-metric space with $s=1$. An example of b-metric is the functional $\rho_p : l_p \times l_p \to \mathbb{R}$, defined by $\rho_p(x,y)=\sum_{i=1}^{\infty}|x_i-y_i|^p$. It is easy to see that in this case $s=2^{p-1}$. Other classical example of b-metric space is $\mathbb{R}$ endowed with the b-metric function $\rho_p(x,y)=|x-y|^p$ for $p\in[1,+\infty)$. It is easy to see that in this case $s=2^{p-1}$ and for $p=1$ we get the metric space of the real numbers with a metric $\rho_1(x,y)=|x-y|$. Another examples of b-metric space are Orlicz and Musielak-Orlicz spaces endowed with Orlicz and Musielak Orlicz function modulars respectively [10, 13].

Definition 1.2. Let (X,ρ) be a b-metric space.

a) A sequence $\{x_n\}_{n=1}^{\infty}$ is called b-convergent if there exists $x\in X$, such that for any $\varepsilon>0$ there exists $N=N(\varepsilon)\in\mathbb{N}$ such that the inequality $\rho(x,x_n)<\varepsilon$ holds true for all $n\ge N$;

b) A sequence $\{x_n\}_{n=1}^{\infty}$ is called b-Cauchy sequence if for any $\varepsilon>0$ there exists $N=N(\varepsilon)\in\mathbb{N}$ such that the inequality $\rho(x_m,x_n)<\varepsilon$ holds true for all $n>m\ge N$;

c) The b-metric space (X,ρ) is called complete b-metric space if any Cauchy sequence is convergent;

d) A subset $A\subseteq X$ is called b-bounded if $\sup\{\rho(x,y): x,y\in A\}<\infty$;

e) If the set A is b-bounded then the number $\sup\left\{\rho(x,y): x,y\in A\right\}$ is called its b-diameter and is denoted with $\delta_b(A)$.

f) A subset $A\subseteq X$ is called b-closed if for any convergent sequence $\left\{x_n\right\}_{n=1}^{\infty}\subset A$ the convergence $\lim\limits_{n\to\infty} x_n = x$ implies $x\in A$.

g) A b-metric function ρ is called continuous if for any $y\in X$ and any $\varepsilon>0$ there exists $\delta=\delta(y,\varepsilon)>0$ such that there holds the inequality $\left|\rho(y,x)-\rho(y,z)\right|<\varepsilon$, provided that $\rho(x,z)<\delta$. It is easy to observe that if ρ is continuous and x_n is b-convergent to x then $\rho(y,x_n)\to\rho(y,x)$.

Every b-convergent sequence in b-metric space is a b-Cauchy sequence. If a sequence is a b-convergent in b-metric space then its limit is unique. In general a b-metric function is not continuous [6, 11].

We will recall the definition of Hardy-Rogers maps in metric spaces.

Definition 1.3. ([8]) Let (X,d) be a metric space. A map $T: X\to X$ is a Hardy-Rogers map is there exist nonnegative constants a_i, $i=1,2,3,4,5$, satisfying $\sum\limits_{i=1}^{5} a_i<1$ such that for each $x,y\in X$ the inequality

$$d(Tx,Ty)\le a_1 d(x,y)+a_2 d(x,Tx)+a_3 d(y,Ty)+a_4 d(x,Ty)+a_5 d(y,Tx)$$

holds.

As pointed in [8] from the symmetry of the function d it follows that $a_2=a_3$ and $a_4=a_5$. Therefore if T is a Hardy-Rogers contraction then there exist $k_1, k_2, k_3\ge 0$, such that $k_1+2k_2+2k_3<1$ and there holds the inequality

(1) $$d(Tx,Ty)\le k_1 d(x,y)+k_2(d(x,Tx)+d(y,Ty))+k_3(d(x,Ty)+d(y,Tx)).$$

For the rest of the article we will consider maps $T:X\to X$ that satisfy (1), where the metric d is replaced with a b-metric ρ.

Classes of Hardy-Rogers map in b-metric space (X,ρ), where the metric d in (1) is replaced with the b-metric ρ, are investigated in [12, 14, 15].

Let us point out that as far as any b-metric is a symmetric function we can assume that for any Hardy-Rogers map, which is defined in a b-metric space (X,ρ), there holds $a_2=a_3$ and $a_4=a_5$.

If $k_1=k_2=0$ and $k_3\in[0,1/2)$ in (1) we get Chatterjea's map [12, 14, 15] in b-metric space. If $k_1=k_3=0$ and $k_2\in[0,1/2)$ in (1) we get Kannan's map [14, 15] in b-metric space. If $k_3=0$ and $k_1+2k_2<1$ in (1) we get Reich's map in b-metric space. Reich's map were introduced in [16] for metric spaces.

We will denote for the rest of the article $\alpha=\dfrac{k_1+k_2}{1-k_2}$, where k_1,k_2,k_3 are the constant from (1), where $k_3=0$. From $k_1+2k_2<1$ it follows that $\alpha\in[0,1)$.

2. FIXED POINTS FOR REICH'S MAPS IN B-METRIC SPACES

Theorem 2.1. Let (X,ρ) be a complete b-metric space, ρ be a continuous function, $T:X\to X$ be a Reich map, such that the inequality $\sup_{n\in\mathbb{N}}\left\{\rho\left(T^n x,x\right)\right\}<\infty$ holds for any $x\in X$. Then

(i) there exists a unique fixed point say ξ of T;

(ii) for any $x_0\in A$ the sequence $\{x_n\}_{n=1}^{\infty}$ converges to ξ, where $x_{n+1}=Tx_n$, $n=0,1,2,\ldots$;

(iii) there holds the a priori error estimate $\rho\left(\xi, T^m x\right) \le \alpha^m \sup_{j\in\mathbb{N}} \rho\left(T^j x, x\right)$.

Lemma 2.2. Let (X,ρ) be a b-metric space and let $T : X \to X$ be a Reich map. Then for any $x \in X$ there holds the inequality

(2) $$\rho\left(T^{n+1}x, T^n x\right) \le \left(\frac{k_1+k_2}{1-k_2}\right)\rho\left(T^n x, T^{n-1}x\right)$$

for any $n > m \ge 1$.

Proof. For any $x \in X$ we get the inequality

$$\rho\left(T^{n+1}x, T^n x\right) \le k_1\rho\left(T^n x, T^{n-1}x\right) + k_2\left(\rho\left(T^{n+1}x, T^n x\right) + \rho\left(T^n x, T^{n-1}x\right)\right)$$

and consequently the inequality (2) holds true.

Corollary 2.3. Let (X,ρ) be a b-metric space and let $T : X \to X$ be a Reich map. Then for any $x \in X$ there holds the inequality

(3) $$\rho\left(T^{n+1}x, T^n x\right) \le \left(\frac{k_1+k_2}{1-k_2}\right)^n \rho(Tx, x)$$

for any $n > m \ge 1$.

Proof. After applying (2) n-times we get

$$\rho\left(T^{n+1}x, T^n x\right) \le \left(\frac{k_1+k_2}{1-k_2}\right)\rho\left(T^n x, T^{n-1}x\right) \le \left(\frac{k_1+k_2}{1-k_2}\right)^2 \rho\left(T^{n-1}x, T^{n-2}x\right) \le \cdots \le \left(\frac{k_1+k_2}{1-k_2}\right)^n \rho(Tx, x).$$

Lemma 2.4. Let (X,ρ) be a b-metric space and let $T : X \to X$ be a Reich map, such that the inequality $\sup_{n\in\mathbb{N}}\left\{\rho\left(T^n x, x\right)\right\} < \infty$. Then for any $x \in X$ and any $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that the inequality $\rho\left(T^n x, T^m x\right) < \varepsilon$ holds for any $n > m \ge N$.

Proof. After applying (1) and (2) we get

$$\begin{aligned}
\rho\left(T^n x, T^m x\right) &\leq k_1\rho\left(T^{n-1}x, T^{m-1}x\right)+k_2\left(\rho\left(T^n x, T^{n-1}x\right)+\rho\left(T^m x, T^{m-1}x\right)\right)\\
&\leq k_1\rho\left(T^{n-1}x, T^{m-1}x\right)+k_2\left(\alpha^{n-1}\rho\left(Tx, x\right)+\alpha^{m-1}\rho\left(Tx, x\right)\right)\\
&\leq k_1\rho\left(T^{n-1}x, T^{m-1}x\right)+k_2\left(\alpha^{n-1}+\alpha^{m-1}\right)\rho\left(Tx, x\right)\\
&\leq \left(k_1\right)^2\rho\left(T^{n-2}x, T^{m-2}x\right)+\left(k_1k_2\left(\alpha^{n-2}+\alpha^{m-2}\right)+k_2\left(\alpha^{n-1}+\alpha^{m-1}\right)\right)\rho\left(Tx, x\right)\\
&= \left(k_1\right)^2\rho\left(T^{n-2}x, T^{m-2}x\right)+k_2\left(k_1\left(\alpha^{n-2}+\alpha^{m-2}\right)+\alpha^{n-1}+\alpha^{m-1}\right)\rho\left(Tx, x\right)
\end{aligned}.$$

After applying the above technique $\left(m-2\right)$-times and using the easy to check inequality $k_1 \leq \alpha$ and the equality $k_2 \cdot \frac{1+\alpha}{\alpha-k_1}=1$ we get

$$\begin{aligned}
\rho\left(T^n x, T^m x\right) &\leq \left(k_1\right)^m\rho\left(T^{n-m}x, x\right)+k_2\left(\sum_{j=0}^{m-1}\left(k_1\right)^j\left(\alpha^{n-1-j}+\alpha^{m-1-j}\right)\right)\rho\left(Tx, x\right)\\
&\leq \left(k_1\right)^m\rho\left(T^{n-m}x, x\right)+k_2\left(\sum_{j=0}^{m-1}\left(k_1\right)^j\alpha^{m-1-j}+\alpha^{n-m}\sum_{j=0}^{m-1}\left(k_1\right)^j\alpha^{m-1-j}\right)\rho\left(Tx, x\right)\\
&\leq \left(k_1\right)^m\rho\left(T^{n-m}x, x\right)+k_2\frac{\alpha^m-\left(k_1\right)^m}{\alpha-k_1}\left(1+\alpha^{n-m}\right)\rho\left(Tx, x\right)\\
&\leq \left(\left(k_1\right)^m\left(1-k_2\frac{1+\alpha}{\alpha-k_1}\right)+k_2\frac{1+\alpha}{\alpha-k_1}\alpha^m\right)\sup_{n\in\mathbb{N}}\left\{\rho\left(T^n x, x\right)\right\}\\
&\leq \alpha^m\sup_{n\in\mathbb{N}}\left\{\rho\left(T^n x, x\right)\right\}
\end{aligned} \tag{4}$$

.

Let $\varepsilon > 0$ be arbitrary chosen. From $\alpha \in (0,1)$ it follows that there exists $N \in \mathbb{N}$, such that there holds the inequality $\alpha^m < \frac{\varepsilon}{\sup_{n\in\mathbb{N}}\left\{\rho\left(T^n x, x\right)\right\}}$ holds for any $n \geq N$. From (4) we get that the inequality $\rho\left(T^n x, T^m x\right) \leq \alpha^m \sup_{n\in\mathbb{N}}\left\{\rho\left(T^n x, x\right)\right\} < \varepsilon$ holds for any $n > m \geq N$.

Proof. of Theorem 2.1 (i) Let $x \in X$ be arbitrary.

From Lemma 2.4 we have that the sequence $\left\{T^n x\right\}_{n=1}^{\infty}$ is a

Cauchy sequence. From the assumption that X is complete b-metric space it follows that the sequence $\left\{T^n x\right\}_{n=1}^{\infty}$ is b-convergent. Therefore it follows that there exists $\xi = \lim_{n\to\infty} T^n x \in X$. After taking a limit on $m \to \infty$ from the assumption that the b-metric is continuous and using that T is Reich map and using Corollary 2.3 we get the inequality

$$\rho(T\xi,\xi) = \lim_{m\to\infty} \rho\left(T\xi, T^m x\right) \le \lim_{m\to\infty}\left(k_1\rho\left(\xi, T^{m-1}x\right) + k_2\left(\rho(T\xi,\xi) + \rho\left(T^m x, T^{m-1}x\right)\right)\right) = k_2\rho(T\xi,\xi)$$

and therefore $\rho(T\xi,\xi) = 0$ i.e. ξ is a fixed point for T.

Let suppose that there are two fixed points $\xi \ne \eta$. Then from the inequality

$$\rho(\xi,\eta) = \rho(T\xi,T\eta) \le k_1\rho(\xi,\eta) + k_2\left(\rho(T\xi,\xi) + \rho(T\eta,\eta)\right) = k_1\rho(\xi,\eta)$$

and the assumption that $k_1 \in [0,1)$ it follows that $\xi = \eta$.

(ii) The proof follows from (i), because any sequence $\left\{T^n x_0\right\}_{n=1}^{\infty}$ is convergent to the fixed point of T, which is unique.

(iii) Let $x \in X$ be arbitrary. From (4) we have that the inequality

$$\rho\left(T^n x, T^m x\right) \le \alpha^m \sup_{n\in\mathbb{N}}\left\{\rho\left(T^n x, x\right)\right\}$$

holds for every $n > m \ge 1$ and every $x \in X$. From (ii) it follows that the sequence $\left\{T^n x\right\}_{n=1}^{\infty}$ converges to the unique fixed point ξ. Therefore using the continuity of ρ we get

$$\rho\left(\xi, T^m x\right) = \lim_{n\to\infty} \rho\left(T^n x, T^m x\right) \le \alpha^m \sup_{n\in\mathbb{N}}\left\{\rho\left(T^n x, x\right)\right\}.$$

As far as any metric space is a b-metric space, then Theorem 2.1 holds true for arbitrary metric space. If (X,d) is a complete metric space and T be Reich map, then the a priori error estimate can be obtained from the proof of (Theorem 1 [8])

(5) $$d\left(\xi, T^m x\right) \le \frac{\alpha^m}{1-\alpha} d\left(Tx, x\right).$$

If we assume that $\sup_{j\in\mathbb{N}} \rho\left(T^j x, x\right) \le \rho\left(Tx, x\right)$ then we will get from Theorem 2.1 the a priori estimate

(6) $$\rho\left(\xi, T^m x\right) \le \alpha^m \rho\left(Tx, x\right).$$

Let us mention that in this case the a priori estimate (5) is better, than (6). Let $\varepsilon \in \left(0, \rho\left(Tx, x\right)\right)$, $m_\alpha \in \mathbb{N}$ be the smallest number, that satisfies (5) and $n_\alpha \in \mathbb{N}$ be the smallest number, that satisfies (6). Then

$$n_\alpha - m_\alpha \ge \left| \frac{\log \frac{\varepsilon(1-\alpha)}{\rho(Tx,x)}}{\log \alpha} \right| - \left(\left| \frac{\log \frac{\varepsilon}{\rho(Tx,x)}}{\log \alpha} \right| + 1 \right) = \left| \frac{\log(1-\alpha)}{\log \alpha} \right| - 1.$$

If $k_1 + 2k_2$ gets close to 1 then α gets closer to 1 and therefore $n_\alpha - m_\alpha$ gets closer to infinity.

We would like to point out that if the space is a metric space than using the triangle inequality we can obtain (5) from Theorem 2.1 iii).

Example 2.5. Let us consider the b-metric space $\left(\mathbb{R}, \rho_p\right)$ for $p \ge 1$. Let us define the map $T:[0,+\infty) \to [0,+\infty)$, by $Tx = \begin{cases} \frac{1}{2}, & x \in [0,1) \\ \frac{1}{4}, & x \in [1,+\infty) \end{cases}$, which is a variation of the examples from [12]. It is easy to observe that the Picard iteration sequence $x_n = Tx_{n-1}$ converges to the fixed point $x = 0$ for any initial point $x_1 \in [0,+\infty)$ (Figure 1).

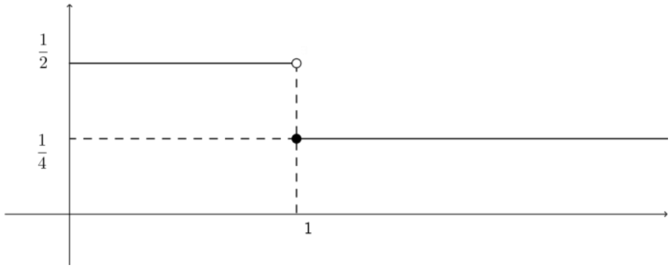

Figure 1

If $x, y \in [0, \beta)$ or $x, y \in [\beta, +\infty)$, then T satisfies the condition in Definition 1.4 for any k_1, k_2, because $\rho_p(Tx, Ty) = | Tx - Ty |^p = 0$.

If $y \in [0,1)$ and $x \in [1, +\infty)$, then we get $\rho_p(Tx, Ty) = \left| \frac{1}{4} - \frac{1}{2} \right|^p = \left| \frac{1}{4} \right|^p$.

Let us denote $f_p(x, y) = |x - y|^p + \left| \frac{1}{4} - x \right|^p + \left| \frac{1}{2} - y \right|^p$. The inequality $f_p(x, y) \geq f_p(1, y)$ holds for any $x \in [1, +\infty)$ and $y \in [0,1)$. From

$$f_p(1, y) = \begin{cases} (1-y)^p + \left(\frac{3}{4} \right)^p + \left(\frac{1}{2} - y \right)^p, & 0 \leq y \leq \frac{1}{2} \\ (1-y)^p + \left(\frac{3}{4} \right)^p + \left(y - \frac{1}{2} \right)^p, & \frac{1}{2} \leq y \leq 1 \end{cases}$$

it is easy to calculate that

$$\min_{y \in [0,1]} f_p(1, y) = \min \left\{ \min_{y \in [0,1/2]} f_p(1, y), \min_{y \in [1/2,1]} f_p(1, y) \right\} = f_p\left(1, \frac{3}{4}\right).$$

Consequently we get

$$\frac{1}{4}\rho_p(x,y)+\frac{1}{4}\Big(\rho_p(Tx,x)+\rho_p(Ty,y)\Big)=\frac{1}{4}f_p(x,y)\geq\frac{1}{4}f_p\left(1,\frac{3}{4}\right)>\left|\frac{1}{4}\right|^p=\rho_p(Tx,Ty)$$

.

Therefore T satisfies the condition in Definition 1.4 for $k_1=k_2=\dfrac{1}{4}$.

When applying fixed point theorems for approximating of a solution of the equation $Tx=x$ we usually find an initial starting point x_0, which belongs to a neighborhood U of the solution ξ, such that $T:U\to U$ and U is bounded and closed. Thus the next Corollary can be applied in a wide class of problems.

Corollary 2.6. Let (X,ρ) be a complete b-metric space, ρ be a continuous function, $A\subseteq X$ be a b-bounded and b-closed set, $T:A\to A$ be Reich map. Then

(i') there exists a unique fixed point say ξ of T;

(ii') for any $x_0\in A$ the sequence $\{x_n\}_{n=1}^{\infty}$ converges to ξ, where $x_{n+1}=Tx_n$, $n=0,1,2,\dots$;

(iii') there holds the a priori error estimate $\rho\left(\xi,T^n x\right)\leq\alpha^m\delta_b\left(A\right)$.

We would like to pose an open question: Is it possible to obtain a similar result for Hardy-Rogers maps in b-metric spaces, without involving the b-metric constant, as like as Theorem 2.1 and [11].

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