ВЪРХУ ЛОКАЛНАТА СХОДИМОСТ НА "*INVERSE WDK*" МЕТОДА

ГЮРХАН Х. НЕДЖИБОВ

ON LOCAL CONVERGENCE ANALYSIS OF THE INVERSE WDK METHOD*

GYURHAN H. NEDZHIBOV

ABSTRACT: *In this paper we study a new modification of the well known Durand-Kerner iterative method for simultaneous approximation of polynomial zeros. We establish new local convergence theorems with error estimates. The main results are motivated by the convergence analysis of the Weierstrass' method provided by Niell in [1]. We also compare the two corresponding radiuses of convergence.*

KEYWORDS*: Polynomial zeros, Simultaneous method, Durand-Kerner method, Weierstrass algorithm, Inverse Weierstrass method, Local convergence.*

1 INTRODUCTION

Let $P(z)$ be a monic polynomial

(1) $$P(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} + z^n,$$

of degree $n \ge 2$, with simple real or complex zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$, and let $z_1^{(0)}, z_2^{(0)}, \ldots, z_n^{(0)}$ be distinct reasonable close approximations of these zeros.

In this study we consider a simultaneous iterative method defined by

(2) $$\mathbf{z}^{(k+1)} = \mathbf{G}\left(\mathbf{z}^{(k)}\right) = \mathbf{G}^{k+1}\left(\mathbf{z}^{(0)}\right), \; k = 0,1,2,\ldots,$$

where $\mathbf{G} : \mathbf{C}^n \to \mathbf{C}^n$ is a vector valued function with components

(3) $$G_i = G_i(\mathbf{z}) = \frac{z_i^2}{z_i + W_i(\mathbf{z})}, \; \mathbf{z} = (z_1, \ldots, z_n), \; i = 1, \ldots, n,$$

and the term

(4) $$W_i(\mathbf{z}) = \frac{P(z_i)}{\prod_{j \ne i}^{n} (z_i - z_j)}, \; (i = 1, \ldots, n)$$

is the so-called *Weierstrass' correction*.

The iteration method (2)-(3) is a modification of the famous Weierstrass' iterative method for simultaneously finding all the zeros of polynomials

(5) $$z_i^{(k+1)} = z_i^{(k)} - W_i(\mathbf{z}^{(k)}), \quad i = 1,2,\ldots,n, \; k \ge 0,$$

which is also called *Durand-Kerner*, *Weierstrass-Dochev*, or shorter the *WDK method*. It is originally proposed by Weierstrass in 1891 [2], rediscovered later by Durand [3], Dochev [4], Kerner [5], Prešić [6], and since then it has been investigated by many authors (see [7, 8, 9, 10,

* This paper is supported by Shumen University under Grant № РД-08-69/02.02.2016г.

11, 12, 13, 14, 15, 16]). The modified method (2)-(3) was firstly introduced in [17], and so called *Inverse WDK method.* Some recent results were obtained in [18, 19].

Throughout this paper, we will use only the norm defined by

$$\|z\|_\infty = \max_i |z_i|,$$

and we will follow the usual convention that a summation over the empty set of indices equals 0, while a product over the same set equals 1.

2 LOCAL CONVERGENCE ANALYSIS

First, we prove some auxiliary results.

Lemma 1 *Let* $c \geq 0$, $d > 2c$, $q > 1$, $n \geq 2$ *and*

(6) $$\left(\frac{d-c}{d-2c}\right)^{n-1} < q.$$

Then the following relations hold true

(7) $$(i)\ c < kd,\ \text{where}\ k = \frac{q-1}{2q-1};$$

(8) $$(ii)\ \frac{c}{d-c} < \frac{q-1}{q}.$$

Proof. (i) The inequality (6) is equivalent to the following two inequalities

(9) $$\frac{d-c}{d-2c} < q^{\frac{1}{n-1}}$$

and consequently

(10) $$c < \frac{q^{\frac{1}{n-1}} - 1}{2q^{\frac{1}{n-1}} - 1} d.$$

The assertion (7)

$$c < \frac{q-1}{2q-1} d$$

follows from the choice of q, assumption $n \geq 2$ and the relation (10).

(ii) The claim (8) follows from (7), the choice of d and the inequality

$$\frac{c}{d-c} < \frac{kd}{d-kd} = \frac{k}{1-k}.$$

Corrolary 1 *Let* $q \in (1,2)$, *then from the relations (6), (7) and (8) of Lemma 1 it follows that*

(11) $$c < \frac{d}{3}$$

and

(12) $$\frac{c}{d-c}\left(\frac{d-c}{d-2c}\right)^{n-1} < 1.$$

Lemma 2 *Let the initial assumptions and (6), (7), (8) of Lemma 1 hold true and let* $q \in (1,2)$. *Then* $q = 4/3$ *is the maximum value of the parameter* q *such that*

(13) $$g(c,d,n) = \frac{\frac{d}{d-c}\left(\frac{d-c}{d-2c}\right)^{n-1} - 1}{1 - \frac{c}{d-c}\left(\frac{d-c}{d-2c}\right)^{n-1}} < 1.$$

Proof. From the left side of (13)

$$g(c,d,n) = \frac{\frac{d}{d-c}\left(\frac{d-c}{d-2c}\right)^{n-1} - 1}{1 - \frac{c}{d-c}\left(\frac{d-c}{d-2c}\right)^{n-1}} = \frac{\left(\frac{d-c}{d-2c}\right)^{n-1} + \frac{c}{d-c}\left(\frac{d-c}{d-2c}\right)^{n-1} - 1}{1 - \frac{c}{d-c}\left(\frac{d-c}{d-2c}\right)^{n-1}}$$

and the relations (6), (7) and (8) it follows

$$g(c,d,n) < \frac{q + \frac{q-1}{q}q - 1}{1 - \frac{q-1}{q}q} = \frac{2(q-1)}{2-q} = h(q).$$

The first derivative of $h(q)$ is $h'(q) = 2/(2-q)^2 > 0$, i.e. $h(q)$ is monotone in the interval $(1,2)$ and also

$$\lim_{q\to 2} h(q) = \infty.$$

But the value $q = 4/3$ is the only solution of the equation $h(q) = 1$, which proves the lemma.

Further, in proving of the main result we will use the identity (given by [9])

(14) $$\prod_{j=1}^{n-1}\frac{u_n - v_j}{u_n - u_j} - 1 = \sum_{s=1}^{n-1}\frac{u_s - v_s}{u_n - u_s}\prod_{j=1}^{s-1}\frac{u_n - v_j}{u_n - u_j},$$

which is valid for any $2n$ numbers u_i, v_i, such that $u_i \neq u_j$ for $i \neq j$ $(i, j = 1,\ldots,n)$.

Theorem 1 *Let* $P \in \mathbf{C}[z]$ *be a monic polynomial of degree* $n \geq 2$, *where*

$$\alpha = \{\alpha \in \mathbf{C}^n : \alpha_i \neq 0 \text{ and } \alpha_i \neq \alpha_j \text{ for } i, j = 1,\ldots,n\}$$

is the root vector of P, *and let*

$$d = \min\{\delta,\gamma\},\ \text{where}\ \delta = \min_{j\neq i}|\alpha_i - \alpha_j|\ \text{and}\ \gamma = \min_i|\alpha_i|.$$

If the initial guess $\mathbf{z}^{(0)} \in \mathbf{C}^n$, *satisfies the inequality*

(15) $$\left\|\mathbf{z}^{(0)} - \alpha\right\| < \rho(n,d) := \frac{q^{\frac{1}{n-1}} - 1}{2q^{\frac{1}{n-1}} - 1}d,\ \text{where}\ q = 4/3,$$

then

(i) the modified Durand-Kerner iteration (2)-(3) is well defined and converges to α *quadratically;*

(ii) the asymptotic convergence rate satisfies

$$\overline{\lim}_{k\to\infty} \frac{\left\| \mathbf{z}^{(k+1)} - \alpha \right\|}{\left\| \mathbf{z}^{(k)} - \alpha \right\|^2} \le \frac{n}{d}. \tag{16}$$

Proof. The $(k+1)^{th}$ iteration stage of the algorithm (2) is

$$z_i^{(k+1)} = \frac{(z_i^{(k)})^2}{z_i^{(k)} + W_i(z^{(k)})}, \quad i = 1,2,\ldots,n, \ k \ge 0. \tag{17}$$

For easy of later comparisons, we will use the following equivalent form of (17) (see [19])

$$z_i^{(k+1)} = z_i^{(k)} - \frac{W_i(z^{(k)})}{1 + \frac{W_i(z^{(k)})}{z_i^{(k)}}}, \quad i = 1,2,\ldots,n, \ k \ge 0 \tag{18}$$

which implies

$$z_i^{(k+1)} - \alpha_i = z_i^{(k)} - \alpha_i - \frac{W_i(z^{(k)})}{1 + \frac{W_i(z^{(k)})}{z_i^{(k)}}}$$

and consequently

$$z_i^{(k+1)} - \alpha_i = (z_i^{(k)} - \alpha_i)\left[1 - \frac{\prod_{j\ne i}^{n} \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}}}{1 + \frac{W_i(z^{(k)})}{z_i^{(k)}}}\right] = (z_i^{(k)} - \alpha_i)\left[\frac{1 - \prod_{j\ne i}^{n} \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} + \frac{W_i(z^{(k)})}{z_i^{(k)}}}{1 + \frac{W_i(z^{(k)})}{z_i^{(k)}}}\right].$$

For the error in each component we get

$$| z_i^{(k+1)} - \alpha_i | = | z_i^{(k)} - \alpha_i | \left| \frac{1 - \prod_{j\ne i}^{n} \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} + \frac{W_i(z^{(k)})}{z_i^{(k)}}}{1 + \frac{W_i(z^{(k)})}{z_i^{(k)}}} \right|,$$

which implies

$$\left\| z^{(k+1)} - \alpha \right\| \le \left\| z^{(k)} - \alpha \right\| \max_i \left| \frac{1 - \prod_{j\ne i}^{n} \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} + \frac{W_i(z^{(k)})}{z_i^{(k)}}}{1 + \frac{W_i(z^{(k)})}{z_i^{(k)}}} \right|. \tag{19}$$

Further, we will bound the amplification factor for the i^{th} component. Let for fixed k and i denote

$$A_i^{(k)} := \left| \frac{1 - \prod_{j\neq i}^{n} \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} + \frac{W_i(z^{(k)})}{z_i^{(k)}}}{1 + \frac{W_i(z^{(k)})}{z_i^{(k)}}} \right|.$$

Then the following inequality is valid

$$(20) \qquad A_i^{(k)} \le \frac{\left| \prod_{j\neq i}^{n} \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} - 1 \right| + \left| \frac{z_i^{(k)} - \alpha_i}{z_i^{(k)}} \right| \left| \prod_{j\neq i}^{n} \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} \right|}{1 - \left| \frac{z_i^{(k)} - \alpha_i}{z_i^{(k)}} \right| \left| \prod_{j\neq i}^{n} \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} \right|}.$$

From (10), (11) and (15) we establish the following inequalities

$$(21) \qquad | z_i^{(k)} - z_j^{(k)} | \ge | \alpha_i - \alpha_j | - | z_i^{(k)} - \alpha_i | - | z_j^{(k)} - \alpha_j | \ge d - 2\left\| z^{(k)} - \alpha \right\| > 0,$$

$$(22) \qquad | z_i^{(k)} | \ge | \alpha_i | - | z_i^{(k)} - \alpha_i | \ge d - \left\| z^{(k)} - \alpha > 0 \right\|$$

and

$$(23) \qquad \left| \frac{z_i^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} \right| = \left| 1 + \frac{z_j^{(k)} - \alpha_j}{z_i^{(k)} - z_j^{(k)}} \right| \le 1 + \frac{\left\| z^{(k)} - \alpha \right\|}{d - 2\left\| z^{(k)} - \alpha \right\|} = \frac{d - \left\| z^{(k)} - \alpha \right\|}{d - 2\left\| z^{(k)} - \alpha \right\|}.$$

Substituting

$$\begin{cases} u_s = z_s^{(k)}, & 1 \le s < i-1 \\ u_s = z_{s+1}^{(k)}, & i \le s \le n-1 \\ u_s = z_i^{(k)}, & s = n \end{cases} \quad \text{and} \quad \begin{cases} v_s = \alpha_s, & 1 \le s < i-1 \\ v_s = \alpha_{s+1}, & i \le s \le n-1 \\ v_s = \alpha_i, & s = n \end{cases}$$

in the identity (14) and using (20), (21), (22) and (23) we obtain

$$A_i^{(k)} \le \frac{\sum_{s=1}^{n-1} \frac{\left\| z^{(k)} - \alpha \right\|}{d - 2\left\| z^{(k)} - \alpha \right\|} \prod_{j=1}^{s-1} \left(\frac{d - \left\| z^{(k)} - \alpha \right\|}{d - 2\left\| z^{(k)} - \alpha \right\|} \right) + \frac{\left\| z^{(k)} - \alpha \right\|}{d - \left\| z^{(k)} - \alpha \right\|} \prod_{j\neq i}^{n} \left(\frac{d - \left\| z^{(k)} - \alpha \right\|}{d - 2\left\| z^{(k)} - \alpha \right\|} \right)}{1 - \frac{\left\| z^{(k)} - \alpha \right\|}{d - \left\| z^{(k)} - \alpha \right\|} \prod_{j\neq i}^{n} \left(\frac{d - \left\| z^{(k)} - \alpha \right\|}{d - 2\left\| z^{(k)} - \alpha \right\|} \right)},$$

which implies

$$(24) \qquad A_i^{(k)} \le \frac{\frac{\left\| z^{(k)} - \alpha \right\|}{d - 2\left\| z^{(k)} - \alpha \right\|} \sum_{s=0}^{n-2} \left(\frac{d - \left\| z^{(k)} - \alpha \right\|}{d - 2\left\| z^{(k)} - \alpha \right\|} \right)^s + \frac{\left\| z^{(k)} - \alpha \right\|}{d - \left\| z^{(k)} - \alpha \right\|} \left(\frac{d - \left\| z^{(k)} - \alpha \right\|}{d - 2\left\| z^{(k)} - \alpha \right\|} \right)^{n-1}}{1 - \frac{\left\| z^{(k)} - \alpha \right\|}{d - \left\| z^{(k)} - \alpha \right\|} \left(\frac{d - \left\| z^{(k)} - \alpha \right\|}{d - 2\left\| z^{(k)} - \alpha \right\|} \right)^{n-1}}$$

and consequently

$$A_i^{(k)} \le \frac{\left(\dfrac{d-\left\|z^{(k)}-\alpha\right\|}{d-2\left\|z^{(k)}-\alpha\right\|}\right)^{n-1} - 1 + \dfrac{\left\|z^{(k)}-\alpha\right\|}{d-\left\|z^{(k)}-\alpha\right\|}\left(\dfrac{d-\left\|z^{(k)}-\alpha\right\|}{d-2\left\|z^{(k)}-\alpha\right\|}\right)^{n-1}}{1-\dfrac{\left\|z^{(k)}-\alpha\right\|}{d-\left\|z^{(k)}-\alpha\right\|}\left(\dfrac{d-\left\|z^{(k)}-\alpha\right\|}{d-2\left\|z^{(k)}-\alpha\right\|}\right)^{n-1}}.$$

Finally, from the last expression, it follows that the amplification factor for the error norm in (19) can be bounded as follows:

$$A^{(k)} := \max_i A_i^{(k)} \le \frac{\dfrac{d}{d-\left\|z^{(k)}-\alpha\right\|}\left(\dfrac{d-\left\|z^{(k)}-\alpha\right\|}{d-2\left\|z^{(k)}-\alpha\right\|}\right)^{n-1} - 1}{1-\dfrac{\left\|z^{(k)}-\alpha\right\|}{d-\left\|z^{(k)}-\alpha\right\|}\left(\dfrac{d-\left\|z^{(k)}-\alpha\right\|}{d-2\left\|z^{(k)}-\alpha\right\|}\right)^{n-1}} = \xi_k(n,d).$$

Substituting $c = \left\|z^{(k)}-\alpha\right\|$ in Lemma 2 and using the assumption (15) and (13) we obtain that

$$\xi_k(n,d) < 1,$$

whenever

$$\left\|z^{(k)}-\alpha\right\| < \rho(n,d).$$

Since this bound does not depend on k and it follows by induction from the assumption (15). Then $\xi_0 < 1$ is a uniform upper bound for all the ξ_k. This completes the proof of the claim (i).

(ii) From (19) and the derivation of (24), it follows that

$$\frac{\left\|z^{(k+1)}-\alpha\right\|}{\left\|z^{(k)}-\alpha\right\|^2} \le \frac{\dfrac{1}{d-2\left\|z^{(k)}-\alpha\right\|}\displaystyle\sum_{s=0}^{n-2}\left(\dfrac{d-\left\|z^{(k)}-\alpha\right\|}{d-2\left\|z^{(k)}-\alpha\right\|}\right)^{s} + \dfrac{1}{d-\left\|z^{(k)}-\alpha\right\|}\left(\dfrac{d-\left\|z^{(k)}-\alpha\right\|}{d-2\left\|z^{(k)}-\alpha\right\|}\right)^{n-1}}{1-\dfrac{\left\|z^{(k)}-\alpha\right\|}{d-\left\|z^{(k)}-\alpha\right\|}\left(\dfrac{d-\left\|z^{(k)}-\alpha\right\|}{d-2\left\|z^{(k)}-\alpha\right\|}\right)^{n-1}}$$

and giving in the limit, as $k \to \infty$, we get $\left\|z^{(k)}-\alpha\right\| \to 0$ and the claim (16).

In the next theorem we consider the case $a_0 = 0$ in (1), i.e. there exists s such that $\alpha_s = 0$ for $s = 1,\ldots,n$. Without loss of generality we can assume that $\alpha_n = 0$.

Theorem 2 *Let* $P \in \mathbf{C}[z]$ *be a monic polynomial of degree* $n \ge 2$ *with simple zeros, such that*

$$\alpha_i \ is \begin{cases} \neq 0, & i = 1,\ldots,n-1 \\ = 0, & i = n \end{cases}$$

and let $d = \{min_{j\neq i} |\alpha_i - \alpha_j| : i, j = 1,\ldots,n\}$. *If the initial guess* $\mathbf{z}^{(0)} \in \mathbf{C}^n$, *satisfies the initial condition (15) from Theorem 1*

$$\left\|\mathbf{z}^{(0)}-\alpha\right\| \le \rho(n,d),$$

the inverse WDK iteration (2)-(3) is well defined and converges

(i) quadratically to $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$*;*

(ii) at least linearly to $\alpha_n = 0$.

Proof. (i) The quadratic convergence and the same asymptotic convergence rate (16) for $i = 1, \ldots, n-1$ can be proved in a similar way that we proved Theorem 1 (i).

(ii) Let consider the function $G_n(z)$, defined by (3) as a function of one variable z_n:

$$\theta(z_n) = G_n(\mathbf{z}) = \frac{z_n^2}{z_n + W_n(\mathbf{z})},$$

then it is easily seen that the point $z_n = 0$ is a fixed point of the function $\theta(z_n)$, i.e. $\theta(0) = 0$ (see [19]).

Let us suppose that $z_n^{(k)} \neq 0$. From (17) and taking into account that $\alpha_n = 0$, we have

$$z_n^{(k+1)} = z_n^{(k)} \left[\frac{1}{1 + \dfrac{W_n(z^{(k)})}{z_n^{(k)}}} \right] = z_n^{(k)} \left[\frac{1}{1 + \prod_{j=1}^{n-1} \dfrac{z_n^{(k)} - \alpha_j}{z_n^{(k)} - z_j^{(k)}}} \right].$$

Then, it follows

$$| z_n^{(k+1)} | = | z_n^{(k)} | \left| \frac{1}{1 + \prod_{j=1}^{n-1} \dfrac{z_n^{(k)} - \alpha_j}{z_n^{(k)} - z_j^{(k)}}} \right| \leq | z_n^{(k)} | \hat{A}_n^{(k)}, \tag{25}$$

where

$$\hat{A}_n^{(k)} = \left[\frac{1}{2 - \left| \prod_{j=1}^{n-1} \dfrac{z_n^{(k)} - \alpha_j}{z_n^{(k)} - z_j^{(k)}} - 1 \right|} \right].$$

Substituting u_i by $z_i^{(k)}$ and v_i by α_i (for $i = 1, \ldots, n$) in (14) and using the relations (21) and (23), it is easy to prove that

$$\left| \prod_{j=1}^{n-1} \frac{z_n^{(k)} - \alpha_j}{z_n^{(k)} - z_j^{(k)}} - 1 \right| \leq \sum_{s=1}^{n-1} \frac{\left\| z^{(k)} - \alpha \right\|}{d - 2\left\| z^{(k)} - \alpha \right\|} \prod_{j=1}^{s-1} \left(\frac{d - \left\| z^{(k)} - \alpha \right\|}{d - 2\left\| z^{(k)} - \alpha \right\|} \right),$$

which implies

$$\left| \prod_{j=1}^{n-1} \frac{z_n^{(k)} - \alpha_j}{z_n^{(k)} - z_j^{(k)}} - 1 \right| \leq \frac{\left\| z^{(k)} - \alpha \right\|}{d - 2\left\| z^{(k)} - \alpha \right\|} \sum_{s=0}^{n-2} \left(\frac{d - \left\| z^{(k)} - \alpha \right\|}{d - 2\left\| z^{(k)} - \alpha \right\|} \right)^s,$$

and consequently

(26) $$\left|\prod_{j=1}^{n-1}\frac{z_n^{(k)}-\alpha_j}{z_n^{(k)}-z_j^{(k)}}-1\right| \le \left(\frac{d-\left\|z^{(k)}-\alpha\right\|}{d-2\left\|z^{(k)}-\alpha\right\|}\right)^{n-1}-1.$$

From (26), substituting $c=\left\|z^{(k)}-\alpha\right\|$ in (10) and using the assumptions of the theorem we obtain

(27) $$\left|\prod_{j=1}^{n-1}\frac{z_n^{(k)}-\alpha_j}{z_n^{(k)}-z_j^{(k)}}-1\right| \le q-1<1.$$

Then the factor $\hat{A}_n^{(k)}$ defined by (25) can be bounded as follows

$$\hat{A}_n^{(k)}<1.$$

Thus we have proved the relation

$$|z_n^{(k+1)}|<|z_n^{(k)}|,\quad k=0,1,2,\dots,$$

i.e. $|z_n^{(k)}|\to 0$, as $k\to\infty$. This completes the proof.

3 COMPARISON WITH A CONVERGENCE RESULT OF WDK-METHOD

The study of local convergence of Weierstrass' method presented by Niell in [1] shows that if the initial approximation $z^{(0)}\in \mathbb{C}^n$ satisfies

(28) $$\left\|\mathbf{z}^{(0)}-\alpha\right\| \le \tau(n,d):=\frac{\sqrt[n-1]{2}-1}{2\sqrt[n-1]{2}-1}d,$$

where $d=\min\{|\alpha_i-\alpha_j|: i\ne j, i,j=1,\dots,n\}$, then the iteration (5) converges to α quadratically, and the asymptotic convergence rate satisfies

(29) $$\overline{\lim}_{k\to\infty}\frac{\left\|\mathbf{z}^{(k+1)}-\alpha\right\|}{\left\|\mathbf{z}^{(k)}-\alpha\right\|^2}\le\frac{n-1}{d}.$$

The radius of convergence $\rho(n,d)$ defined by (15) (in Theorem 1) corresponds to the radius $\tau(n,\delta)$ defined by (28) with the same asymptotic rate (16). It is obvious that the radius $\rho(n,d)$ is smaller from $\tau(n,d)$. Indeed, the ratio $\rho(n,d)/\tau(n,d)$ does not depend on d, and the limit

$$\lim_{n\to\infty}\frac{\rho(n,d)}{\tau(n,d)}\approx 0.41.$$

The comparative values of the asserted radiuses are included in Table 1.

Table 1: Some comparative values of $\rho(n,d)$ and $\tau(n,d)$.

n	$\frac{\rho(n,d)}{d}$	$\frac{\tau(n,d)}{d}$	$\frac{\rho(n,d)}{\tau(n,d)}$
1	0.2	0.33	0.6
2	0.11	0.22	0.52
3	0.08	0.17	0.48

4	0.06	0.13	0.47
5	0.05	0.11	0.46
10	0.03	0.06	0.43
20	0.01	0.03	0.42
50	0.005	0.013	0.42
100	0.002	0.006	0.41

REFERENCES

[1] **A.M. Niel**, The simultaneous approximation of polynomial roots, Computers and Mathematics with applications, 41 (2001) 1–14.
[2] **K. Weierstrass**, Neuer Beweis des satzes, dass jede ganze rationale function einer Veränderlichen dargestellt werden kann als ein product aus linearen functionen derselben Veränderlichen, Sitzungsber. K. Preuss. Akad. Wiss. (Berlin) (1891) 1085–1101.
[3] **E. Durand**, Solutions Numériques des Équations Algébriques, Tome 1: Équations du Type F(x)=0, Racines d'un Polynôme, Masson, Paris, 1960.
[4] **K. Dochev**, Modified Newton methodfor the simultaneous computation of all roots of a given algebraic equation, Phys. Math. J. Bulg. Acad. Sci. 5 (1962) 136–139 (in Bulgarian).
[5] **I.O. Kerner**, Ein gesamtschrittverfahren zur Berechnung der Nullstellen von Polynomen, Numer. Math. 8 (1966) 290–294.
[6] **S.B. Preši** c′ , Un procéd é itératif pour la factorisation des polynômes, C. R. Acad. Sci. Paris Sér. A 262 (1966) 862–863.
[7] **S.B. Preši** c′ , A convergence theorem for a method for simultaneous determination of all zeros of a polynomial, Publ. Inst. Math. (N.S.) 28 (1980) 159–-165.
[8] **S.M. Zheng**, On convergence of the Durand-Kerners method for finding all roots of a polynomial simultaneously, Kexue Tongbao 27 (1982) 1262-–1265.
[9] **N.V. Kjurkchiev**, S.M. Markov, Two interval methods for algebraic equations with real roots, Pliska Stud. Math. Bulg., 5 , (1983) 118–131.
[10] **D.R. Wang** and F.G. Zhao, The theory of Smales point estimation and its applications, J. Comput. Appl. Math. 60 (1995) 253–269.
[11] **M. Petkovi** c′ , C. Carstensen and M. Trajkovi c′ , Weierstrass formula and zero-finding methods, Numer. Math. 69 (1995) 353–372.
[12] **M. Petkovi** c′ , On initial conditions for the convergence of simultaneous root-finding methods, Computing 57 (1996) 163–177.
[13] **D.F. Han**, The convergence of the Durand-Kerner method for simultaneously finding all zeros of a polynomial, J. Comput. Math. 18 (2000) 567–570.
[14] **P.D. Proinov**, A new semilocal convergence theorem for the Weierstrass method from data at one point, C. R. Acad. Bulg. Sci 59, No 2 (2006) 131–136.
[15] **P.D. Proinov**, M.D. Petkova, Convergence of the Weierstrass method for simultaneous approximation of polynomial zeros, Comptes rendus de l'Académie bulgare des Sciences, Tome 66, No 6 (2013).
[16] **P.D. Proinov**, General convergence theorems for iterative processes and applications to the Weierstrass root-finding method, arXiv:1503.05243, (2015).
[17] **G.H. Nedzhibov**, Similarity transformations between some companion matrices, Application of Mathematics in Engineering and Economics (AMEE14), AIP Conf. Proc., 1631, (2014) 375–382.
[18] **G.H. Nedzhibov**, Inverse Weierstrass-Durand-Kerner Iterative Method, International Journal of Applied Mathematics, ISSN:2051-5227, Vol.28, Issue.2, (2013) 1258–1264.
[19] **G.H. Nedzhibov**, Convergence of the modified inverse Weierstrass method for simultaneous approximation of polynomial zeros, Communications in Numerical Analysis, Volume 2016, No. 1 (2016), 74-80.