PROPERTIES OF MULTIPLICATIVE INTEGRALS I *

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ABSTRACT: *In this paper we present some important properties of a special kind of multiplicative integrals connected with semigroups of operators T_t ($t \geq 0$ and $t \leq 0$) with unbounded K^r- generators iA, presented as regular couplings of dissipative and antidissipative operators in a Hilbert space H. These properties play an important role in the course of obtaining the asymptotics of the corresponding nondissipatve cueves $T_t f$ as $t \to \pm\infty$ ($f \in H$).*

KEYWORDS: *Nonselfadjoint operator, dissipative operator, unbounded operator, operator colligation, triangular model, coupling, multiplicative integral*

In this paper we present properties of special kind of multiplicative integrals which play an important role in the further development of the investigations of nonselfadjoint unbounded operators (with finite dimensional imaginary parts) based on the theory of the characteristic functions and the triangular models of M.S. Livšic. The presented properties are inequalities concerning the multiplicative integrals which are so-called limit values of multiplicative integrals connected with nonselfadjoint unbounded K^r- operators A in a Hilbert space H with differen domains of A and its adjoint A^* and presented as a regular coupling of dissipative and antidissipative operators with real absolutely continuous spectra. The triangular model of the regular couplings of this class of nonselfadjoint operators has been introduced and investigated by K.P. Kirchev and G.S. Borisova in [3, 4, 5]. In the course of investigations of these operators A (the characteristic operator functions, the resolvent of A, the asymptotic behaviour of the corresponding continuous curves) we use the properties of the multiplicative integrals from the form

$$(1) \qquad \overset{\rightarrow}{\int_a^b} e^{-i\frac{1+\lambda\alpha(v)}{\alpha(v)-\lambda}T(v)dv},$$

($\lambda \in \mathbb{C}$, $\lambda \neq \alpha(v)$ for $v \in [a,b]$), where $\alpha(v)$ is a nondecreasing real function in $[a,b]$, $T(v)$ is a measurable nonnegative $m \times m$ matrix function, satisfying the conditions

$$(2) \qquad \int_a^b trT(v)dv < +\infty; \qquad \int_a^b ||T(v)||dv < +\infty.$$

The integral (1) is the multiplicative Stieltjes integral, defined as

$$(3) \qquad \overset{\rightarrow}{\int_a^b} e^{f(t)G(t)dt} = \lim_{\max\triangle\theta_k\to 0} \overset{\rightarrow}{\prod_{k=1}^{n}} e^{f(\tau_k)(E(\theta_k)-E(\theta_{k-1}))} = \\ = e^{f(\tau_1)(E(\theta_1)-E(\theta_0))}e^{f(\tau_2)(E(\theta_2)-E(\theta_1))}\dots e^{f(\tau_n)(E(\theta_n)-E(\theta_{n-1}))},$$

where $E(\theta) = \int_a^\theta G(t)dt$ and the limit in (3) is taken over all the partitions $a = \theta_0 < \theta_1 < \ldots < \theta_n = b$ of the interval $[a,b]$ and all the choices of intermediate points τ_k such

*Partially supported by Scientific Research Grant RD-08-75/2016 of Shumen University

that $\theta_{k-1} \leq \tau_k \leq \theta_k$ $(k = 1, 2, \dots, n)$, $G(\theta)$ is integrable matrix function on $[a, b]$ and $||E(\theta') - E(\theta'')|| \leq |\theta' - \theta''|$.

Further to avoid the complications of writing we will consider the multiplicative integral (1) in the case when $\alpha(v) = v$. The general case of $\alpha(v)$ can be considered analogously.

The limits of the multiplicative integrals from the form (in the sense of a strong limit)

$$s - \lim_{\delta\to 0} \overset{\overset{b}{\curvearrowright}}{\underset{a}{\int}} e^{\frac{-iT(v)}{v-(x\pm i\delta)}dv} = \tag{4}$$
$$= s - \lim_{\delta\to 0} \overset{x-\delta}{\underset{a}{\overrightarrow{\int}}} e^{\frac{-iT(v)}{v-x}dv} e^{\pm\pi T(x)} \overset{b}{\underset{x+\delta}{\overrightarrow{\int}}} e^{\frac{-iT(v)}{v-x}dv}$$

are used essentially in the case of bounded dissipative operators [7], the case of bounded nonselfadjoint operators, presented as a coupling of dissipative and antidisipative operators [2], the case of unbounded nonselfadjoint operators, presented as a coupling of dissipative and antidissipative operators and with equal domains of the operator and its adjoint [4]. The equality (4) is proved by L.A. Sakhnovich in [7] and it is an analogue for the multiplicative integrals of the well-known Privalov's theorem [6] for the limit values for the integral

$$f(\lambda) = \int_a^b \frac{p(t)}{t-\lambda} dt$$

in the scalar case.

In the case of considered unbounded K^r- operators in the course of obtaining the asymptotics of the corresponding nondissipative curves we essentially use the existence and the form of the limit values of the multiplicative integrals (in the sense of a strong limit) for almost all $x \in [a, b]$

$$s - \lim_{\delta\to 0} \overset{b}{\underset{a}{\overrightarrow{\int}}} e^{-i\frac{1+(x\pm i\delta)v}{v-(x\pm i\delta)}T(v)dv}, \quad \delta > 0. \tag{5}$$

At first we will remind a proposition, obtained in [7], and the form of the limit (5), presented and obtained in [4], [5], [1], which we will use in this paper.

Theorem 1. *([7]) Let $m \times m$ matrix functions $(m \leq \infty)$ $B_1(t)$ and $B_2(t)$ are integrable in $[a, b]$ and for almost all t in $[a, b]$ satisfy the inequalities*

$$\frac{B_k(t) - B^*(t)}{i} \leq 0.$$

Then

$$\left\| \overset{b}{\underset{a}{\overrightarrow{\int}}} e^{-iB_1(t)dt} - \overset{b}{\underset{a}{\overrightarrow{\int}}} e^{-iB_2(t)dt} \right\| \leq \int_a^b ||B_1(t) - B_2(t)||dt.$$

Theorem 2. *([1], [4], [5]) Let the matrix function $T(x)$ is integrable and nonnegative in $[a, b]$. Then for almost all $x \in \mathbb{R}$ there exist the limits (5) and they have the form*

$$s - \lim_{\delta\to 0} \overset{b}{\underset{a}{\overrightarrow{\int}}} e^{-i\frac{1+(x\pm i\delta)v}{v-(x\pm i\delta)}T(v)dv} = \tag{6}$$
$$= s - \lim_{\varepsilon\to 0} \overset{x-\varepsilon}{\underset{a}{\overrightarrow{\int}}} e^{-i\frac{1+vx}{v-x}T(v)dv} e^{\pm\pi(1+x^2)T(x)} \overset{b}{\underset{x+\varepsilon}{\overrightarrow{\int}}} e^{-i\frac{1+vx}{v-x}T(v)dv}$$

($\delta > 0$, $\varepsilon > 0$).

Let $m \times m$ matrix function $T(x)$, defined on $\mathbb{R}$, satisfy the conditions:

(i) $||T(x)|| \leq C$, $||xT(x)|| \leq C$ $\forall x \in \mathbb{R}$;

(ii) $T(x) \in C_{\alpha_1}(\mathbb{R})$, $xT(x) \in C_{\alpha_2}(\mathbb{R})$ $(0 < \alpha_1 \leq 1, 0 < \alpha_2 \leq 1)$ (i.e. $||T(x_1) - T(x_2)|| \leq C|x_1 - x_2|^{\alpha_1}$, $||x_1T(x_1) - x_2T(x_2)|| \leq C|x_1 - x_2|^{\alpha_2}$ $\forall x_1, x_2 \in \mathbb{R}$).

Here C is a constant and we denote by $||\ \ ||$ the norm in $\mathbb{C}^m$. Let now $\alpha = \min\{\alpha_1, \alpha_2\}$. Then the next inequalities hold.

Theorem 3. *Let the nonpositive or nonnegative matrix function $T(x)$ be integrable matrix function on $\mathbb{R}$ and satisfy the conditions **(i)** and **(ii)**. Then*

$$||e^{iT(x)(1+x^2)\ln(x-\xi)} - e^{iT(\xi)(1+\xi^2)\ln(x-\xi)}|| \leq \widetilde{C}(1+|x|)|x-\xi|^{\alpha'} \tag{7}$$

for some constant $\widetilde{C} > 0$, for all $\alpha' : 0 < \alpha' < \alpha$, and for x, ξ such that $0 < x - \xi < 1$.

Proof. At first for the case $x - \xi \geq 1$ we have

$$\begin{gathered} ||e^{iT(x)(1+x^2)\ln(x-\xi)} - e^{iT(\xi)(1+\xi^2)\ln(x-\xi)}|| \leq \\ \leq ||e^{iT(x)(1+x^2)\ln(x-\xi)}|| + ||e^{iT(\xi)(1+\xi^2)\ln(x-\xi)}|| \leq 2 \leq 2(x-\xi)^{\alpha'} \end{gathered} \tag{8}$$

for all $\alpha' > 0$.

Next we consider the case $0 < x - \xi < 1$. Then using Theorem 1 after straightforward calculations we obtain

$$\begin{gathered} ||e^{iT(x)(1+x^2)\ln(x-\xi)} - e^{iT(\xi)(1+\xi^2)\ln(x-\xi)}|| = \\ = \left\| e^{-iT(x)(1+x^2)\int\limits_x^{\xi+1} \frac{1}{v-\xi}dv} - e^{-iT(\xi)(1+\xi^2)\int\limits_x^{\xi+1} \frac{1}{v-\xi}dv} \right\| = \\ = \left\| \overset{\rightarrow}{\int\limits_x^{\xi+1}} e^{\frac{-iT(x)(1+x^2)}{v-\xi}}dv - \overset{\rightarrow}{\int\limits_x^{\xi+1}} e^{\frac{-iT(\xi)(1+\xi^2)}{v-\xi}}dv \right\| \leq \int\limits_x^{\xi+1} \left\| \frac{T(x)(1+x^2) - T(\xi)(1+\xi^2)}{v-\xi} \right\| dv \leq \\ \leq \widehat{C}(1+|x|)|x-\xi|^{\alpha} \int\limits_x^{\xi+1} \frac{1}{|v-\xi|}dv = \widehat{C}(1+|x|)|x-\xi|^{\alpha}|\ln(x-\xi)| \leq \\ \leq \widehat{C}(1+|x|)|x-\xi|^{\alpha}\frac{1}{\beta e}|x-\xi|^{-\beta} \end{gathered} \tag{9}$$

(for all $\beta > 0$) where we have used the next inequality:

$$||T(x)(1+x^2) - T(\xi)(1+\xi^2)|| \leq \widehat{C}(1+|x|)|x-\xi|^{\alpha}.$$

In the right hand side of the last inequality of the relations (9) we have used the inequality

$$|\ln y| \leq \frac{1}{\beta e}y^{-\beta} \quad \forall \beta > 0, \quad 0 < y < 1. \tag{10}$$

Now the relations (8) and (9) imply that the inequality (7) is true and the theorem is proved. □

Theorem 4. *Let the nonpositive or nonnegative matrix function $T(x)$ be integrable matrix function on $\mathbb{R}$ and satisfy the conditions **(i)** and **(ii)**. Then for each $\alpha' : 0 < \alpha' < \alpha$ there exists a constant $\widetilde{C} > 0$ such that*

$$||e^{iT(\xi)(1+\xi^2)\ln(\xi-w)} - e^{iT(x)(1+x^2)\ln(x-w)}|| \leq \widetilde{C}(1+|x|)\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} \tag{11}$$

for all $w, \xi, x : w < \xi < x$, $0 < x - w < 1$.

Proof. We consider the first case

$$w < \xi < \frac{w+x}{2}.$$

This implies that $\frac{x-\xi}{\xi-w} > 1$. Then we have

$$\begin{aligned} &||e^{iT(\xi)(1+\xi^2)\ln(\xi-w)} - e^{iT(x)(1+x^2)\ln(x-w)}|| \leq \\ &\leq ||e^{iT(\xi)(1+\xi^2)\ln(\xi-w)}|| + ||e^{iT(x)(1+x^2)\ln(x-w)}|| = 2 \leq \left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} \end{aligned} \tag{12}$$

for all $\alpha' : 0 < \alpha' < 1$.

Now we consider the second case

$$\frac{w+x}{2} \leq \xi < x.$$

Hence $\frac{x-\xi}{\xi-w} \leq 1$. Now we obtain

$$\begin{aligned} &||e^{iT(\xi)(1+\xi^2)\ln(\xi-w)} - e^{iT(x)(1+x^2)\ln(x-w)}|| = \\ &= ||e^{iT(\xi)\ln(\xi-w)}e^{iT(\xi)\xi^2\ln(\xi-w)} - e^{iT(x)\ln(x-w)}e^{iT(x)x^2\ln(x-w)}|| = \\ &= \left\| e^{iT(x)\int\limits_{x-1}^{w}\frac{1}{v-x}dv} e^{iT(x)x^2\int\limits_{x-1}^{w}\frac{1}{v-x}dv} - e^{iT(\xi)\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} e^{iT(\xi)\xi^2\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} \right\| \leq \\ &\leq \left\| e^{iT(x)\int\limits_{x-1}^{w}\frac{1}{v-x}dv} e^{iT(x)x^2\int\limits_{x-1}^{w}\frac{1}{v-x}dv} - e^{iT(x)\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} e^{iT(x)x^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} \right\| + \\ &+ \left\| e^{iT(x)\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} e^{iT(x)x^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} - e^{iT(\xi)\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} e^{iT(\xi)\xi^2\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} \right\|. \end{aligned} \tag{13}$$

Then we consider the second addend in the right hand side of the relations (13) and after straightforward calculations we obtain consecutively

$$\begin{aligned} &\left\| e^{iT(x)\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} e^{iT(x)x^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} - e^{iT(\xi)\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} e^{iT(\xi)\xi^2\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} \right\| = \\ &= \left\| e^{iT(x)\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} e^{iT(x)x^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} - e^{iT(\xi)\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} e^{iT(x)x^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} + \right. \\ &+ e^{iT(\xi)\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} e^{iT(x)x^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} - e^{iT(\xi)\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} e^{iT(x)x^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} + \\ &+ e^{iT(\xi)\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} e^{iT(x)x^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} - e^{iT(\xi)\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} e^{iT(\xi)\xi^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} + \\ &\left. + e^{iT(\xi)\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} e^{iT(\xi)\xi^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} - e^{iT(\xi)\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} e^{iT(\xi)\xi^2\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} \right\| \leq \\ &\leq \left\| e^{iT(x)\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} - e^{iT(\xi)\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} \right\| \left\| e^{iT(x)x^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} \right\| + \\ &+ \left\| e^{iT(\xi)\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} - e^{iT(\xi)\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} \right\| \left\| e^{iT(x)x^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} \right\| + \end{aligned}$$

$$\begin{aligned}&+\left\|e^{iT(\xi)\int_{\xi-1}^{w}\frac{1}{v-\xi}dv}\right\|\left\|e^{iT(x)x^2\int_{\xi-1}^{w}\frac{1}{v-x}dv}-e^{iT(\xi)\xi^2\int_{\xi-1}^{w}\frac{1}{v-x}dv}\right\|+\\ &+\left\|e^{iT(\xi)\int_{\xi-1}^{w}\frac{1}{v-\xi}dv}\right\|\left\|e^{iT(\xi)\xi^2\int_{\xi-1}^{w}\frac{1}{v-x}dv}-e^{iT(\xi)\xi^2\int_{\xi-1}^{w}\frac{1}{v-\xi}dv}\right\|=\end{aligned}\tag{14}$$

$$=\left\|e^{iT(x)\int_{\xi-1}^{w}\frac{1}{v-x}dv}-e^{iT(\xi)\int_{\xi-1}^{w}\frac{1}{v-x}dv}\right\|+\left\|e^{iT(\xi)\int_{\xi-1}^{w}\frac{1}{v-x}dv}-e^{iT(\xi)\int_{\xi-1}^{w}\frac{1}{v-\xi}dv}\right\|+$$
$$+\left\|e^{iT(x)x^2\int_{\xi-1}^{w}\frac{1}{v-x}dv}-e^{iT(\xi)\xi^2\int_{\xi-1}^{w}\frac{1}{v-x}dv}\right\|+\left\|e^{iT(\xi)\xi^2\int_{\xi-1}^{w}\frac{1}{v-x}dv}-e^{iT(\xi)\xi^2\int_{\xi-1}^{w}\frac{1}{v-\xi}dv}\right\|.$$

From the assumptions **(i)** for the matrix function $T(x)$ we have that

$$\|T(\xi)\xi^2\|\le|\xi|.\|T(\xi)\xi\|\le C|\xi|\le C(|\xi-x|+|x|).\tag{15}$$

Then for the last addend in the right hand side of the relations (14) we obtain (applying Theorem 1)

$$\begin{aligned}&\left\|e^{iT(\xi)\xi^2\int_{\xi-1}^{w}\frac{1}{v-x}dv}-e^{iT(\xi)\xi^2\int_{\xi-1}^{w}\frac{1}{v-\xi}dv}\right\|=\left\|\overset{\rightarrow}{\int_{\xi-1}^{w}}e^{iT(\xi)\xi^2\frac{1}{v-x}dv}-\overset{\rightarrow}{\int_{\xi-1}^{w}}e^{iT(\xi)\xi^2\frac{1}{v-\xi}dv}\right\|\le\\ &\le\int_{\xi-1}^{w}\|T(\xi)\xi^2\tfrac{1}{v-x}-T(\xi)\xi^2\tfrac{1}{v-\xi}\|dv=\|T(\xi)\xi^2\|\int_{\xi-1}^{w}\left(\tfrac{1}{\xi-v}-\tfrac{1}{x-v}\right)dv\le\\ &\le C(|\xi-x|+|x|)\int_{\xi-1}^{w}\left(\tfrac{1}{\xi-v}-\tfrac{1}{x-v}\right)dv=\\ &=C(|\xi-x|+|x|)(-\ln(\xi-w)+\ln(\xi-\xi+1)+\ln(x-w)-\ln(x-\xi+1))=\\ &=C(|\xi-x|+|x|)\left(\ln\tfrac{x-w}{\xi-w}-\ln(x-\xi+1)\right)=\\ &=C(|\xi-x|+|x|)\left(\ln\left(1+\tfrac{x-\xi}{\xi-w}\right)-\ln(1+(x-\xi))\right)\le\\ &\le C(|\xi-x|+|x|)\left(\tfrac{x-\xi}{\xi-w}+x-\xi\right)\le\\ &\le C(1+|x|)\left(\left(\tfrac{x-\xi}{\xi-w}\right)^{\alpha'}+(x-\xi)^{\alpha'}\right)\end{aligned}\tag{16}$$

(for all $\alpha':0<\alpha'<1$), where we have used the inequality

$$\ln(1+y)\le y\qquad\forall y\ge0.\tag{17}$$

Consequently,

$$\left\|e^{iT(\xi)\xi^2\int_{\xi-1}^{w}\frac{1}{v-x}dv}-e^{iT(\xi)\xi^2\int_{\xi-1}^{w}\frac{1}{v-\xi}dv}\right\|\le C(1+|x|)\left(\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'}+(x-\xi)^{\alpha'}\right).\tag{18}$$

For the third addend in the right hand side of the relations (14) using Theorem 1 we obtain

$$\begin{aligned}&\left\|e^{iT(x)x^2\int_{\xi-1}^{w}\frac{1}{v-x}dv}-e^{iT(\xi)\xi^2\int_{\xi-1}^{w}\frac{1}{v-x}dv}\right\|=\left\|\overset{\rightarrow}{\int_{\xi-1}^{w}}e^{iT(x)x^2\frac{1}{v-x}dv}-\overset{\rightarrow}{\int_{\xi-1}^{w}}e^{iT(\xi)\xi^2\frac{1}{v-x}dv}\right\|\le\\ &\le\int_{\xi-1}^{w}\|T(x)x^2-T(\xi)\xi^2\|\tfrac{1}{|v-x|}dv.\end{aligned}\tag{19}$$

But from the assumptions **(i)** and **(ii)** for the matrix function $T(x)$ it follows that

$$\|T(x)x^2 - T(\xi)\xi^2\| \leq C(1+|x|)(x-\xi)^{\alpha}. \tag{20}$$

Now using (20) we continue the inequalities (19) and we obtain consecutively the next relations

$$\begin{gathered} \left\| e^{iT(x)x^2 \int\limits_{\xi-1}^{w} \frac{1}{v-x}dv} - e^{iT(\xi)\xi^2 \int\limits_{\xi-1}^{w} \frac{1}{v-x}dv} \right\| \leq \int\limits_{\xi-1}^{w} \|T(x)x^2 - T(\xi)\xi^2\| \tfrac{1}{|v-x|}dv \leq \\ \leq C(1+|x|)(x-\xi)^{\alpha} \int\limits_{\xi-1}^{w} \tfrac{1}{x-v}dv = C(1+|x|)(x-\xi)^{\alpha}(\ln(1+x-\xi) - \ln(x-w)) = \\ = C(1+|x|)(x-\xi)^{\alpha}\left(\ln(1+x-\xi) + |\ln(\xi-w)| + \left|\ln\left(1+\tfrac{x-\xi}{\xi-w}\right)\right|\right) \leq \\ \leq C(1+|x|)(x-\xi)^{\alpha}\left((x-\xi) + \tfrac{1}{\beta e}\tfrac{1}{(\xi-w)^{\beta}} + \tfrac{x-\xi}{\xi-w}\right) < \\ < C(1+|x|)(x-\xi)^{\alpha}\left(2 + \tfrac{1}{\beta e}\tfrac{1}{(x-\xi)^{\beta}}\right) < C(1+|x|)\left(2+\tfrac{1}{\beta e}\right)(x-\xi)^{\alpha-\beta} = \\ = \widehat{C}(1+|x|)(x-\xi)^{\alpha-\beta} \end{gathered} \tag{21}$$

for some constant $\widehat{C} = C\left(2+\frac{1}{\beta e}\right)$ and for all $\beta:\ 0<\beta<\alpha<1$. In the course of obtaining the inequalities we have used the inequalities (17) and (10).

Now using again Theorem 1 and the inequalities (17) and (10) for the second addend in the right hand side of the last inequality in (14) we obtain

$$\begin{gathered} \left\| e^{iT(\xi) \int\limits_{\xi-1}^{w} \frac{1}{v-x}dv} - e^{iT(\xi) \int\limits_{\xi-1}^{w} \frac{1}{v-\xi}dv} \right\| \leq \int\limits_{\xi-1}^{w} \|T(\xi)\| \left|\tfrac{1}{v-x} - \tfrac{1}{v-\xi}\right| dv \leq \\ \leq C \int\limits_{\xi-1}^{w} \left|\tfrac{1}{v-x} - \tfrac{1}{v-\xi}\right| dv = C \int\limits_{\xi-1}^{w} \left(\tfrac{1}{\xi-v} - \tfrac{1}{x-v}\right) dv = C\left(\ln\tfrac{x-w}{\xi-w} - \ln(1+x-\xi)\right) \leq \\ \leq C\left(\ln\left(1+\tfrac{x-\xi}{\xi-w}\right) + |\ln(1+x-\xi)|\right) \leq C\left(\tfrac{x-\xi}{\xi-w} + x - \xi\right) \leq C\left(\left(\tfrac{x-\xi}{\xi-w}\right)^{\alpha'} + (x-\xi)^{\alpha'}\right) \end{gathered}$$

$(\forall\alpha' : 0 < \alpha' \leq 1)$. Consequently,

$$\left\| e^{iT(\xi) \int\limits_{\xi-1}^{w} \frac{1}{v-x}dv} - e^{iT(\xi) \int\limits_{\xi-1}^{w} \frac{1}{v-\xi}dv} \right\| \leq C\left((x-\xi)^{\alpha'} + \left(\frac{x-\xi}{\xi-w}\right)^{\alpha'}\right), \quad \forall\alpha' : 0<\alpha'\leq 1. \tag{22}$$

Finally, for the first addend in the right hand side of the last inequality in (14) by analogous calculations we obtain

$$\begin{gathered} \left\| e^{iT(x) \int\limits_{\xi-1}^{w} \frac{1}{v-x}dv} - e^{iT(\xi) \int\limits_{\xi-1}^{w} \frac{1}{v-x}dv} \right\| = \left\| \overset{\rightarrow}{\int\limits_{\xi-1}^{w}} e^{iT(x)\frac{1}{v-x}dv} - \overset{\rightarrow}{\int\limits_{\xi-1}^{w}} e^{iT(\xi)\frac{1}{v-x}dv} \right\| \leq \\ \leq \int\limits_{\xi-1}^{w} \|T(x) - T(\xi)\| \tfrac{1}{|v-x|}dv \leq C|x-\xi|^{\alpha_1} \int\limits_{\xi-1}^{w} \tfrac{1}{x-v}dv \leq \\ \leq C(x-\xi)^{\alpha_1}\left(x-\xi+\tfrac{1}{\beta e}\tfrac{1}{(\xi-w)^{\beta}}\tfrac{x-\xi}{\xi-w}\right) \leq \widehat{C}(x-\xi)^{\alpha_1-\beta} \leq \widehat{C}(x-\xi)^{\alpha-\beta}, \end{gathered} \tag{23}$$

i.e.

$$\left\| e^{iT(x) \int\limits_{\xi-1}^{w} \frac{1}{v-x}dv} - e^{iT(\xi) \int\limits_{\xi-1}^{w} \frac{1}{v-x}dv} \right\| \leq \widehat{C}(x-\xi)^{\alpha-\beta} \quad \forall\beta : 0<\beta<\alpha\leq 1. \tag{24}$$

Consequently, the inequalities (18), (21), (22) and (24) together with (14) imply that

$$\left\| e^{iT(x)\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} e^{iT(x)x^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} - e^{iT(\xi)\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} e^{iT(\xi)\xi^2\int\limits_{\xi-1}^{w}\frac{1}{v-\xi}dv} \right\| \leq \tag{25}$$
$$\leq \widehat{C}(x-\xi)^{\alpha'} + C\left((x-\xi)^{\alpha'} + \left(\frac{x-\xi}{\xi-w}\right)^{\alpha'}\right) + \widehat{C}(1+|x|)(x-\xi)^{\alpha'} +$$
$$+C(1+|x|)\left((x-\xi)^{\alpha'} + \left(\frac{x-\xi}{\xi-w}\right)^{\alpha'}\right) \quad \forall\alpha' : 0<\beta<\alpha'<1.$$

Now for the first addend in the right hand side of the last inequality of (13) straightforward calculations show that

(26)

$$\left\| e^{iT(x)\int\limits_{x-1}^{w}\frac{1}{v-x}dv} e^{iT(x)x^2\int\limits_{x-1}^{w}\frac{1}{v-x}dv} - e^{iT(x)\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} e^{iT(x)x^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} \right\| \leq$$
$$\leq \left\| e^{iT(x)\int\limits_{x-1}^{w}\frac{1}{v-x}dv} e^{iT(x)x^2\int\limits_{x-1}^{w}\frac{1}{v-x}dv} e^{iT(x)\int\limits_{\xi-1}^{x-1}\frac{1}{v-x}dv} \right\| \left\| e^{-iT(x)\int\limits_{\xi-1}^{x-1}\frac{1}{v-x}dv} - e^{iT(x)x^2\int\limits_{\xi-1}^{x-1}\frac{1}{v-x}dv} \right\| \leq$$
$$\leq \left\| e^{-iT(x)\int\limits_{\xi-1}^{x-1}\frac{1}{v-x}dv} - e^{iT(x)x^2\int\limits_{\xi-1}^{x-1}\frac{1}{v-x}dv} \right\| \leq \int\limits_{\xi-1}^{x-1} \left\| \frac{-iT(x)}{v-x} - \frac{T(x)x^2}{v-x} \right\| dv \leq$$
$$\leq \|T(x) + T(x)x^2\| \int\limits_{\xi-1}^{x-1} \frac{1}{|v-x|}dv \leq C(1+|x|)\ln(1+x-\xi) \leq$$
$$\leq C(1+|x|)(x-\xi) \leq C(1+|x|)(x-\xi)^{\alpha'} \quad \forall\alpha' : 0<\alpha'<1,$$

i.e.

$$\left\| e^{iT(x)\int\limits_{x-1}^{w}\frac{1}{v-x}dv} e^{iT(x)x^2\int\limits_{x-1}^{w}\frac{1}{v-x}dv} - e^{iT(x)\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} e^{iT(x)x^2\int\limits_{\xi-1}^{w}\frac{1}{v-x}dv} \right\| \leq C(1+|x|)(x-\xi)^{\alpha'} \tag{27}$$

$(\forall\alpha' : 0<\alpha'<1,\ 0<x-\xi<1)$.

The inequalities (27), (25) and (13) imply that

$$||e^{iT(\xi)(1+\xi^2)\ln(\xi-w)} - e^{iT(x)(1+x^2)\ln(x-w)}|| \leq C(1+|x|)(x-\xi)^{\alpha'} + \tag{28}$$
$$+\widehat{C}(x-\xi)^{\alpha'} + C\left((x-\xi)^{\alpha'} + \left(\frac{x-\xi}{\xi-w}\right)^{\alpha'}\right) + \widehat{C}(1+|x|)(x-\xi)^{\alpha'} +$$
$$+C(1+|x|)\left((x-\xi)^{\alpha'} + \left(\frac{x-\xi}{\xi-w}\right)^{\alpha'}\right)$$

$\forall\alpha' : 0<\beta<\alpha'<\alpha<1$, β is a fixed number satisfying the inequalities $0<\beta<1$. The inequality (28) can be written in the form

$$||e^{iT(\xi)(1+\xi^2)\ln(\xi-w)} - e^{iT(x)(1+x^2)\ln(x-w)}|| \leq \widehat{C}'(1+|x|)\left((x-\xi)^{\alpha'} + \left(\frac{x-\xi}{\xi-w}\right)^{\alpha'}\right) \tag{29}$$

for an appropriate constant $\widehat{C}'$ (which depend on β) and $\forall\alpha' : 0<\beta<\alpha'<\alpha<1$.

Further the inequalities (12) and (29) imply that

$$||e^{iT(\xi)(1+\xi^2)\ln(\xi-w)} - e^{iT(x)(1+x^2)\ln(x-w)}|| \leq 2\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} \tag{30}$$

for all $\alpha' : 0 < \alpha' < 1$ when $w < \xi \leq \frac{w+x}{2}$ and

$$||e^{iT(\xi)(1+\xi^2)\ln(\xi-w)} - e^{iT(x)(1+x^2)\ln(x-w)}|| \leq \widehat{C}'(1+|x|)\left((x-\xi)^{\alpha'} + \left(\frac{x-\xi}{\xi-w}\right)^{\alpha'}\right) \tag{31}$$

$\forall\alpha' : 0 < \beta < \alpha' < \alpha < 1$, β is a fixed number satisfying the inequalities $0 < \beta < 1$ when $w < \frac{w+x}{2} < \xi < x$.

Now the inequalities (30) and (31) can be written in the form

$$||e^{iT(\xi)(1+\xi^2)\ln(\xi-w)} - e^{iT(x)(1+x^2)\ln(x-w)}|| \leq \widetilde{C}(1+|x|)\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} \tag{32}$$

for some constant $\widetilde{C}$, $\forall\alpha' : 0 < \beta < \alpha' < \alpha < 1$, β is a fixed number satisfying the inequalities $0 < \beta < 1$ and for all w, ξ, x such that $w < \xi < x$. The theorem is proved. □

Finally, it have to mention that all presented inequalities in this paper are satisfied when $T(x)$ is nonnegative or nonpositive operator function in infinite dimensional space.

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