PROPERTIES OF MULTIPLICATIVE INTEGRALS II *

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ABSTRACT: *This paper is a continuation of the paper [2] and presents a part of some important properties of a special kind of multiplicative integrals connected with semigroups of operators T_t ($t \geq 0$ and $t \leq 0$) with unbounded K^r- generators iA, presented as regular couplings of dissipative and antidissipative operators in a Hilbert space H. These properties play an important role in the course of obtaining the asymptotics of the corresponding nondissipatve cueves $T_t f$ as $t \to \pm\infty$ ($f \in H$).*

KEYWORDS: *Nonselfadjoint operator, dissipative operator, unbounded operator, operator colligation, triangular model, coupling, multiplicative integral*

This paper is a continuations of the paper [2] and it presents a part of properties of special kind of multiplicative integrals using the obtained results in [2]. These properties of multiplcative integrals play an important role in the further development of the investigations of nonselfadjoint unbounded operators (with finite dimensional imaginary parts) based on the theory of the characteristic functions and the triangular models of M.S. Livšic. The presented properties are inequalities concerning the multiplicative integrals which are so-called limit values of multiplicative integrals connected with nonselfadjoint unbounded K^r- operators A in a Hilbert space H with differen domains of A and its adjoint A^* and presented as a regular coupling of dissipative and antidissipative operators with real absolutely continuous spectra. The triangular model of the regular couplings of this class of nonselfadjoint operators has been introduced and investigated by K.P. Kirchev and G.S. Borisova in [4, 5, 6]. In the course of investigations of these operators A (the characteristic operator functions, the resolvent of A, the asymptotic behaviour of the corresponding continuous curves) we use the properties of the multiplicative integrals from the form

$$(1) \qquad \overset{\curvearrowright}{\int_a^b} e^{-i\frac{1+\lambda\alpha(v)}{\alpha(v)-\lambda}T(v)dv},$$

($\lambda \in \mathbb{C}$, $\lambda \neq \alpha(v)$ for $v \in [a, b]$), where $\alpha(v)$ is a nondecreasing real function in $[a, b]$, $T(v)$ is a measurable nonnegative $m \times m$ matrix function, satisfying the conditions

$$(2) \qquad \int_a^b trT(v)dv < +\infty; \qquad \int_a^b ||T(v)||dv < +\infty.$$

The integral (1) is the multiplicative Stieltjes integral, defined as

$$(3) \qquad \overset{\curvearrowright}{\int_a^b} e^{f(t)G(t)dt} = \lim_{\max\triangle\theta_k\to 0} \overset{\curvearrowright}{\prod_{k=1}^{n}} e^{f(\tau_k)(E(\theta_k)-E(\theta_{k-1}))} = \\ = e^{f(\tau_1)(E(\theta_1)-E(\theta_0))} e^{f(\tau_2)(E(\theta_2)-E(\theta_1))} \dots e^{f(\tau_n)(E(\theta_n)-E(\theta_{n-1}))},$$

where $E(\theta) = \int_a^\theta G(t)dt$ and the limit in (3) is taken over all the partitions $a = \theta_0 < \theta_1 < \ldots < \theta_n = b$ of the interval $[a, b]$ and all the choices of intermediate points τ_k such

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that $\theta_{k-1} \leq \tau_k \leq \theta_k$ $(k = 1, 2, \dots, n)$, $G(\theta)$ is integrable matrix function on $[a, b]$ and $||E(\theta') - E(\theta'')|| \leq |\theta' - \theta''|$.

The limits of the multiplicative integrals from the form (in the sense of a strong limit)

$$s - \lim_{\delta \to 0} \overset{\curvearrowright}{\int_a^b} e^{\frac{-iT(v)}{v-(x\pm i\delta)}dv} = \tag{4}$$
$$= s - \lim_{\delta \to 0} \overset{\curvearrowright}{\int_a^{x-\delta}} e^{\frac{-iT(v)}{v-x}dv} e^{\pm\pi T(x)} \overset{\curvearrowright}{\int_{x+\delta}^{b}} e^{\frac{-iT(v)}{v-x}dv}$$

are used essentially in the case of bounded dissipative operators [8], the case of bounded nonselfadjoint operators, presented as a coupling of dissipative and antidisipative operators [3], the case of unbounded nonselfadjoint operators, presented as a coupling of dissipative and antidissipative operators and with equal domains of the operator and its adjoint [5]. The equality (4) is proved by L.A. Sakhnovich in [8] and it is an analogue for the multiplicative integrals of the well-known Privalov's theorem [7] for the limit values for the integral

$$f(\lambda) = \int_a^b \frac{p(t)}{t - \lambda} dt$$

in the scalar case.

In the case of considered unbounded K^r- operators in the course of obtaining the asymptotics of the corresponding nondissipative curves we essentially use the existence and the form of the limit values of the multiplicative integrals (in the sense of a strong limit) for almost all $x \in [a, b]$

$$s - \lim_{\delta \to 0} \overset{\curvearrowright}{\int_a^b} e^{-i\frac{1+(x\pm i\delta)v}{v-(x\pm i\delta)}T(v)dv}, \quad \delta > 0. \tag{5}$$

At first we will remind a proposition, obtained in [8], and the form of the limit (5), presented and obtained in [5], [6], [1], which we will use in this paper.

Theorem 1. *([8]) Let the $m \times m$ matrix functions $(m \leq \infty)$ $B_1(t)$ and $B_2(t)$ are integrable in $[a, b]$ and for almost all t in $[a, b]$ satisfy the inequalities*

$$\frac{B_k(t) - B^*(t)}{i} \leq 0.$$

Then

$$\left\| \overset{\curvearrowright}{\int_a^b} e^{-iB_1(t)dt} - \overset{\curvearrowright}{\int_a^b} e^{-iB_2(t)dt} \right\| \leq \int_a^b ||B_1(t) - B_2(t)|| dt.$$

Theorem 2. *([1], [5], [6]) Let the matrix function $T(x)$ is integrable and nonnegative in $[a, b]$. Then for almost all $x \in \mathbb{R}$ there exist the limits (5) and they have the form*

$$s - \lim_{\delta \to 0} \overset{\curvearrowright}{\int_a^b} e^{-i\frac{1+(x\pm i\delta)v}{v-(x\pm i\delta)}T(v)dv} = \tag{6}$$
$$= s - \lim_{\varepsilon \to 0} \overset{\curvearrowright}{\int_a^{x-\varepsilon}} e^{-i\frac{1+vx}{v-x}T(v)dv} e^{\pm\pi(1+x^2)T(x)} \overset{\curvearrowright}{\int_{x+\varepsilon}^{b}} e^{-i\frac{1+vx}{v-x}T(v)dv}$$

($\delta > 0$, $\varepsilon > 0$).

Let now $\alpha(x)$ be a nondecreasing unbounded real function, defined in (a,b) $(-\infty \le a < b \le +\infty)$. Let $\Pi(x)$ be a measurable $n\times m$ $(1 \le n \le m, r \le m)$ matrix function whose rows are linearly independent on each point of a set with a positive measure and satisfying the conditions

$$\int_a^b tr\ B(x)dx < +\infty, \quad \int_a^b ||\Pi(x)||^2 dx < +\infty, \tag{7}$$

where $B(x) = \Pi^*(x)\Pi(x)$.

Let $L : \mathbb{C}^m \longrightarrow \mathbb{C}^m$, $L^* = L$, $\det L \ne 0$. Without loss of generality we can suppose that L has the representation

$$L = J_1 - J_2 + S + S^*, \tag{8}$$

where $J_1, J_2, S, S^* : \mathbb{C}^m \longrightarrow \mathbb{C}^m$,

$$J_1 = \begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix},\ J_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r_1} \end{pmatrix},\ S = \begin{pmatrix} 0 & 0 \\ \widehat{S} & 0 \end{pmatrix}, \tag{9}$$

I_k is the identity matrix in $\mathbb{C}^k$ $(k = r_1, m - r_1)$, $\widehat{S}$ is a $(m-r_1)\times r_1$ matrix, r_1 is the number of the positive eigenvalues and $m-r_1$ is the number of the negative eigenvalues of the matrix L.

Let the matrix function $B(x)$ satisfy also the conditions

$$B(x)J_1 = J_1 B(x)$$

and $\alpha(x)B(x)J_2$ be an integrable matrix function on (a,b) $(-\infty \le a < b \le +\infty)$. Let us consider a Hilbert space $\mathbf{L}^2(a,b;\mathbb{C}^n)$, whose elements are vector functions $f(x)$ from the form

$$f(x) = (f_1(x), f_2(x), \dots, f_n(x)), \qquad (f_k \in \mathbf{L}^2(a,b)).$$

The scalar product in $\mathbf{L}^2(a,b;\mathbb{C}^n)$ is defined by the formula

$$(f,g) = \int_a^b f(x)g^*(x)dx.$$

We will denote by $||\ \ ||$ the norm of an operator function in $\mathbb{C}^n$ and by $||\ \ ||_{\mathbf{L}^2}$- the norm in $\mathbf{L}^2(a,b;\mathbb{C}^n)$.

Let $Q(x)$ be a measurable matrix function on (a,b) satisfying the condition

$$\Pi(x)Q(x) = I \tag{10}$$

for almost all $x \in (a,b)$. Then the operators P_1 and P_2, defined by the equalities

$$P_1 f(x) = f(x)\Pi(x)J_1Q(x), \quad P_2 f(x) = f(x)\Pi(x)J_2Q(x)$$

onto $\mathbf{L}^2(a,b;\mathbb{C}^n)$, are orthogonal projectors in $\mathbf{L}^2(a,b;\mathbb{C}^n)$.

The model A, describing the class of K^r-operators presented as a coupling of dissipative and antidissipative operators with real absolutely continuous spectra and with different domains of A and A^* has been introduced in [4] and has the form

$$
\begin{aligned}
&Af(x) = AGg(x) = \alpha(x)g(x) + \\
&+ i\int_a^x g(\xi)(\alpha(\xi)+i)\Pi(\xi)J_1 \overset{\overrightarrow{x}}{\underset{\xi}{\int}} e^{i\alpha(v)B_1(v)dv} J_1\Pi^*(x)(\alpha(x)-i)d\xi - \\
&- i\int_a^x g(\xi)(\alpha(\xi)+i)\Pi(\xi)J_2 \overset{\overrightarrow{x}}{\underset{\xi}{\int}} e^{-i\alpha(v)B_2(v)dv} J_2\Pi^*(x)(\alpha(x)-i)d\xi + \\
&+ \int_a^b g(\xi)(\alpha(\xi)+i)\Pi(\xi)J_2 \overset{\overrightarrow{b}}{\underset{\xi}{\int}} e^{-i\alpha(v)B_2(v)dv} d\xi S \overset{\overrightarrow{x}}{\underset{a}{\int}} e^{B_1(v)dv} J_1\Pi^*(x),
\end{aligned} \tag{11}
$$

where G is an invertible operator in $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ and

$$G = I + P_1KP_2, \tag{12}$$

$$
\begin{aligned}
&KP_2g(x) = \\
&= -i\int_a^b g(\xi)(\alpha(\xi)+i)\Pi(\xi)J_2 \overset{\overrightarrow{b}}{\underset{\xi}{\int}} e^{-i\alpha(v)B_2(v)dv} d\xi S \overset{\overrightarrow{x}}{\underset{a}{\int}} e^{B_1(v)dv} J_1\Pi^*(x),
\end{aligned} \tag{13}
$$

$B_1(x) = B(x)J_1$, $B_2(x) = B(x)J_2$, for each $f(x) = Gg(x) \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ such that $Af \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$. Using the form of the projectors P_1, P_2 the model A, defined by (11), takes the the form

$$Af(x) = AGg(x) = P_1AP_1g(x) + P_2AP_2g(x) + iKP_2g(x), \tag{14}$$

where K is defined by (13) and

$$
\begin{aligned}
&P_1AP_1g(x) = \alpha(x)g(x)\Pi(x)J_1Q(x) + \\
&+ i\int_a^x g(\xi)(\alpha(\xi)+i)\Pi(\xi)J_1 \overset{\overrightarrow{x}}{\underset{\xi}{\int}} e^{i\alpha(v)B_1(v)dv} J_1\Pi^*(x)(\alpha(x)-i)d\xi, \\
&P_2AP_2g(x) = \alpha(x)g(x)\Pi(x)J_2Q(x) - \\
&- i\int_a^x g(\xi)(\alpha(\xi)+i)\Pi(\xi)J_2 \overset{\overrightarrow{x}}{\underset{\xi}{\int}} e^{-i\alpha(v)B_2(v)dv} J_2\Pi^*(x)(\alpha(x)-i)d\xi, \\
&P_1AP_2g(x) = \\
&= \int_a^b g(\xi)(\alpha(\xi)+i)\Pi(\xi)J_2 \overset{\overrightarrow{b}}{\underset{\xi}{\int}} e^{-i\alpha(v)B_2(v)dv} d\xi S \overset{\overrightarrow{x}}{\underset{a}{\int}} e^{B_1(v)dv} J_1\Pi^*(x), \\
&P_2AP_1g(x) = 0
\end{aligned} \tag{15}
$$

and $D_A = G(D_{A_1} \oplus D_{A_2}) \subset \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$. The representation (15) and straightforward calculations show that $A_1 = P_1A$ is a dissipative operator onto $P_1G^{-1}D_A$ and $A_2 = P_2A$ is an antidissipative operator onto $P_2G^{-1}D_A = P_2D_A = D_{A_2}$. In other words

$$
\begin{aligned}
&Im\ (A_1P_1G^{-1}f(x), P_1G^{-1}f(x)) \ge 0, \quad (f \in D_A), \\
&Im\ (A_2P_2G^{-1}f(x), P_2G^{-1}f(x)) \le 0, \quad (f \in D_A).
\end{aligned}
$$

The representation (14) implies that A is a regular coupling of a dissipative operator and an antidissipative one, i.e.

$$A = A_1P_1G^{-1} + A_2P_2 + iKP_2$$

and $A = A_1 \vee A_2$.

The model A, defined by (11), is a closed densely defined operator as a coupling of a dissipative operator and an antidissipative one with real spectra.

For our further considerations we need the resolvent of the coupling of the model A, defined by (11). It turns out that for each $\lambda : Im\ \lambda \neq 0$ the operator $A - \lambda I$ is invertible and the explicit form of the resolvent can be obtained. In the case $\alpha(x) = x$ when $\lambda \neq i$ the resolvent $(A - \lambda I)^{-1}$ is given by the next theorem.

Theorem 3. *([6]) The model A, defined by (11), has the resolvent*

$$(16)\quad \begin{gathered} (A-\lambda I)^{-1} f(x) = \tfrac{f(x)}{\alpha(x)-\lambda} - \\ -i \int_{-\infty}^{x} \tfrac{\alpha(\xi)+i}{\alpha(\xi)-\lambda} f(\xi) \Pi(\xi) J_1 \overset{\curvearrowright}{\int_{\xi}^{x}} e^{-i\frac{1+\lambda\alpha(v)}{\alpha(v)-\lambda} B_1(v)dv} d\xi J_1 \Pi^*(x) \tfrac{\alpha(\xi)-i}{\alpha(\xi)-\lambda} + \\ +i \int_{-\infty}^{x} \tfrac{\alpha(\xi)+i}{\alpha(\xi)-\lambda} f(\xi) \Pi(\xi) J_2 \overset{\curvearrowright}{\int_{\xi}^{x}} e^{i\frac{1+\lambda\alpha(v)}{\alpha(v)-\lambda} B_2(v)dv} d\xi J_2 \Pi^*(x) \tfrac{\alpha(\xi)-i}{\alpha(\xi)-\lambda} - \\ -\tfrac{i}{\lambda-i} \int_{-\infty}^{+\infty} \tfrac{\alpha(\xi)+i}{\alpha(\xi)-\lambda} f(\xi) \Pi(\xi) J_2 \overset{\curvearrowright}{\int_{\xi}^{+\infty}} e^{i\frac{1+\lambda\alpha(v)}{\alpha(v)-\lambda} B_2(v)dv} dx S . \\ \cdot \overset{\curvearrowright}{\int_{-\infty}^{x}} e^{-i\frac{1+\lambda\alpha(v)}{\alpha(v)-\lambda} B_1(v)dv} J_1 \Pi^*(x) \tfrac{\alpha(x)-i}{\alpha(x)-\lambda} \end{gathered}$$

for each $\lambda : Im\ \lambda \neq 0$, $\lambda \neq i$, for each $f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ and the resolvent is a bounded operator in $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$.

The model (11) generates semigroups of operators $\{T_t\}_{t\le 0}$ and $\{T_t\}_{t\ge 0}$ from the class (C_0) with generators iA (see [6], [5]), defined by the equality

$$(17)\quad \begin{gathered} T_t f(x) = -\tfrac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{it(\xi - i\delta)} (A - (\xi - i\delta)I)^{-1} f(x) d\xi + \\ +\tfrac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{it(\xi + i\delta)} (A - (\xi + i\delta)I)^{-1} f(x) d\xi \end{gathered}$$

in the sense of a principal value where $f = (A - \lambda_0 I)^{-1} g$ for all $g \in D_1$, λ_0 is an arbitrary fixed number with $Im\ \lambda_0 > 0$, δ is an arbitrary number with $0 < \delta < Im\ \lambda_0$ when $t > 0$ and $Im\ \lambda_0 < 0$, $0 < \delta < -Im\ \lambda_0$ when $t < 0$.

The explicit obtaining of the asymptotics of the corresponding nondissipative processes $T_t f$ as $t \to \pm\infty$ allows to construct the scattering theory (as in the bounded case of the model A in [3]) for the couple (A^*, A): in other words to obtain the wave operators $W_\pm(A^*, A)$, the scattering operator and the similarity of A and the operator $\mathcal{Q}$ of multiplication by the independent variable. All results are obtained explicitly in [4], [5]), [6] using the multiplicative integrals, their properties and matrix generalization of the classical gamma-function, introduced in [3].

The presented properties of the multiplicative integrals in this paper and in [2] play an important role for obtaining the asymptotics of the corresponding nondissipative processes $T_t f$ as $t \to \pm\infty$.

Let $m \times m$ matrix function $T(x)$, defined on $\mathbb{R}$, satisfy the conditions:

(i) $||T(x)|| \le C$, $||xT(x)|| \le C$ $\forall x \in \mathbb{R}$;

(ii) $T(x) \in C_{\alpha_1}(\mathbb{R})$, $xT(x) \in C_{\alpha_2}(\mathbb{R})$ $(0 < \alpha_1 \leq 1, 0 < \alpha_2 \leq 1)$ (i.e. $||T(x_1) - T(x_2)|| \leq C|x_1 - x_2|^{\alpha_1}$, $||x_1 T(x_1) - x_2 T(x_2)|| \leq C|x_1 - x_2|^{\alpha_2}$ $\forall x_1, x_2 \in \mathbb{R}$).

Here C is a constant and we denote by $||\ \ ||$ the norm in $\mathbb{C}^m$. Let now $\alpha = \min\{\alpha_1, \alpha_2\}$.

Further we will introduce some appropriate denotations. Let us denote the next operators using the multiplicative integrals and the limit values from the form (6):

$$\widetilde{T}(x) = (1 + x^2)T(x), \tag{18}$$

$$\mathcal{F}_w(x \pm i\delta, u) = \overset{\overset{u}{\rightarrow}}{\underset{w}{\int}} e^{-i\frac{1+(x\pm i\delta)v}{v-(x\pm i\delta)}T(v)dv}, \tag{19}$$

$$\mathcal{F}_w^{\pm}(x, u) = s - \lim_{\delta \to 0} \overset{\overset{u}{\rightarrow}}{\underset{w}{\int}} e^{-i\frac{1+(x\pm i\delta)v}{v-(x\pm i\delta)}T(v)dv} \tag{20}$$

for all $w, u, x \in \mathbb{R}$ such that $-\infty \leq w < u \leq +\infty$ and

$$\mathcal{F}_w^{\pm}(x, u) = s - \lim_{\delta \to 0} \overset{\overset{x-\delta}{\rightarrow}}{\underset{w}{\int}} e^{-i\frac{1+vx}{v-x}T(v)dv} e^{\pm\pi\widetilde{T}(x)} \overset{\overset{u}{\rightarrow}}{\underset{x+\delta}{\int}} e^{-i\frac{1+vx}{v-x}T(v)dv}, \tag{21}$$

$$\mathcal{P}_w(x) = \mathcal{F}_w^{+}(x, u) - \mathcal{F}_w^{-}(x, u), \tag{22}$$

$$\mathcal{R}_w^{\pm 1}(x) = s - \lim_{\delta \to 0} \overset{\overset{x-\delta}{\rightarrow}}{\underset{w}{\int}} e^{-i\frac{1+vx}{v-x}T(v)dv} e^{\pm\pi\widetilde{T}(x)} \left(s - \lim_{\delta \to 0} \overset{\overset{x-\delta}{\rightarrow}}{\underset{w}{\int}} e^{-i\frac{1+vx}{v-x}T(v)dv} \right)^{-1}, \tag{23}$$

$$\mathcal{U}_{2w}(x) = s - \lim_{\delta \to 0} \overset{\overset{x-\delta}{\rightarrow}}{\underset{w}{\int}} e^{-i\frac{1+vx}{v-x}T(v)dv} e^{iT(x)\int_w^{x-\delta} \frac{1+vx}{v-x}dv} e^{-iT(x)x(x-\delta-w)}, \tag{24}$$

$$\mathcal{P}_{2w}^{\pm}(x, u) = \mathcal{R}_w^{\pm 1}(x)\mathcal{U}_{2w}(x) e^{i\widetilde{T}(x)\ln\frac{x-w}{u-x}}, \tag{25}$$

$$\mathcal{U}_3(x, u) = \lim_{\delta \to 0} e^{-iT(x)x(u-x-\delta)} e^{iT(x)\int_{x+\delta}^{u} \frac{1+vx}{v-x}dv} \overset{\overset{u}{\rightarrow}}{\underset{x+\delta}{\int}} e^{-i\frac{1+vx}{v-x}T(v)dv}, \tag{26}$$

$$\mathcal{Q}_w^{\pm}(x, u) = \mathcal{P}_{2w}^{\pm}(x, u) e^{-iT(x)x(u-x)} e^{i\widetilde{T}(x)\ln(u-x)} e^{-i\widetilde{T}(u)\ln(u-x)}, \tag{27}$$

$$\mathcal{V}_{-\infty}(x) = \lim_{\delta \to 0} \overset{\overset{x-\delta}{\rightarrow}}{\underset{-\infty}{\int}} e^{-i\frac{1+vx}{v-x}T(v)dv} e^{i\widetilde{T}(x)\ln\delta} \tag{28}$$

for all w, u, x such that $-\infty \leq w < x < u \leq +\infty$.

Then the next inequalities hold.

Theorem 4. *Let the nonnegative or nonpositive matrix function $T(x)$ be integrable matrix function on $\mathbb{R}$ and satisfy the conditions **(i)** and **(ii)**. Then*

$$||\mathcal{F}_w^{\pm}(\xi;x)-\mathcal{Q}_w^{\pm}(\xi;x)||\le \widetilde{C}(1+|x|)(x-\xi)^{\alpha'} \tag{29}$$

for some constant $\widetilde{C}>0$, for all $\alpha':0<\alpha'<\alpha$, and for all $w,\xi,x: w<\xi<x$, $0<x-w<1$.

Proof. At first we will note that from (21), (23), (24), (25), (26) it follows after straightforward calculations that $\mathcal{F}_w^{\pm}(\xi,x)$ from (21) can be written in the form

$$\mathcal{F}_w^{\pm}(\xi,x)=\mathcal{P}_{2w}^{\pm}(\xi,x)\mathcal{U}_3(\xi,x). \tag{30}$$

Then from (30) and (27) we obtain

$$\begin{gathered}\|\mathcal{F}_w^{\pm}(\xi,x)-\mathcal{Q}_w^{\pm}(\xi,x)\|=\\ =\left\|\mathcal{P}_{2w}^{\pm}(\xi,x)\mathcal{U}_3(\xi,x)-\mathcal{P}_{2w}^{\pm}(\xi,x)e^{-iT(\xi)\xi(x-\xi)}e^{i\widetilde{T}(\xi)\ln(x-\xi)}e^{-i\widetilde{T}(x)\ln(x-\xi)}\right\|\le\\ \le\left\|\mathcal{P}_{2w}^{\pm}(\xi,x)\right\|\left\|\mathcal{U}_3(\xi,x)-e^{-iT(\xi)\xi(x-\xi)}e^{i\widetilde{T}(\xi)\ln(x-\xi)}e^{-i\widetilde{T}(x)\ln(x-\xi)}\right\|.\end{gathered} \tag{31}$$

On the other hand

$$\left\|\mathcal{P}_{2w}^{\pm}(\xi,x)\right\|=\left\|\mathcal{R}_w^{\pm1}(\xi)\mathcal{U}_{2w}(\xi)e^{i\widetilde{T}(\xi)\ln\frac{\xi-w}{x-\xi}}\right\|\le\left\|\mathcal{R}_w^{\pm1}(\xi)\right\|\|\mathcal{U}_{2w}(\xi)\|\left\|e^{i\widetilde{T}(\xi)\ln\frac{\xi-w}{x-\xi}}\right\|. \tag{32}$$

But $\|\mathcal{R}_w^{\pm1}(\xi)\|\le 1$, $\|\mathcal{U}_{2w}(\xi)\|\le 1$ and $\left\|e^{i\widetilde{T}(\xi)\ln\frac{\xi-w}{x-\xi}}\right\|=1$ (because $\widetilde{T}(\xi)$ is selfadjoint operator function) and hence

$$\left\|\mathcal{P}_{2w}^{\pm}(\xi,x)\right\|\le 1. \tag{33}$$

On the other hand for the second multiplier in the right hand side of the last inequality in (31) it follows that

$$\begin{gathered}\left\|\mathcal{U}_3(\xi,x)-e^{-iT(\xi)\xi(x-\xi)}e^{i\widetilde{T}(\xi)\ln(x-\xi)}e^{-i\widetilde{T}(x)\ln(x-\xi)}\right\|=\\ =\left\|e^{-iT(\xi)\xi(x-\xi)}\lim_{\delta\to0}e^{iT(\xi)\int\limits_{\xi+\delta}^{x}\frac{1+v\xi}{v-\xi}dv}\overrightarrow{\int\limits_{\xi+\delta}^{u}}e^{-i\frac{1+v\xi}{v-\xi}T(v)dv}-\right.\\ \left.-e^{-iT(\xi)\xi(x-\xi)}e^{i\widetilde{T}(\xi)\ln(x-\xi)}e^{-i\widetilde{T}(x)\ln(x-\xi)}\right\|\le\\ \le\left\|e^{-iT(\xi)\xi(x-\xi)}\right\|\left\|\lim_{\delta\to0}e^{iT(\xi)\int\limits_{\xi+\delta}^{x}\frac{1+v\xi}{v-\xi}dv}\overrightarrow{\int\limits_{\xi+\delta}^{u}}e^{-i\frac{1+v\xi}{v-\xi}T(v)dv}-e^{i\widetilde{T}(\xi)\ln(x-\xi)}e^{-i\widetilde{T}(x)\ln(x-\xi)}\right\|\le\\ \le\left\|\lim_{\delta\to0}\left(e^{iT(\xi)\int\limits_{\xi+\delta}^{x}\frac{1+v\xi}{v-\xi}dv}-\overleftarrow{\int\limits_{\xi+\delta}^{u}}e^{i\frac{1+v\xi}{v-\xi}T(v)dv}\right)\overrightarrow{\int\limits_{\xi+\delta}^{u}}e^{-i\frac{1+v\xi}{v-\xi}T(v)dv}\right\|+\\ +\left\|e^{i\widetilde{T}(\xi)\ln(x-\xi)}-e^{i\widetilde{T}(x)\ln(x-\xi)}\right\|\left\|e^{-i\widetilde{T}(x)\ln(x-\xi)}\right\|\le\\ \le\left\|\lim_{\delta\to0}\left(e^{iT(\xi)\int\limits_{\xi+\delta}^{x}\frac{1+v\xi}{v-\xi}dv}-\overleftarrow{\int\limits_{\xi+\delta}^{u}}e^{i\frac{1+v\xi}{v-\xi}T(v)dv}\right)\overrightarrow{\int\limits_{\xi+\delta}^{u}}e^{-i\frac{1+v\xi}{v-\xi}T(v)dv}\right\|+\\ +\widetilde{C}'(1+|x|)(x-\xi)^{\alpha'}\end{gathered} \tag{34}$$

($\forall x, \xi : 0 < x - \xi < 1$, $\alpha' : 0 < \alpha' < \alpha < 1$). In the right hand side of the last inequality of the relations (34) we have used Theorem 3 in [2].

Further, for the first addend in the right hand side of (34) applying Theorem 1 we obtain consecutively

$$\left\| \left(e^{iT(\xi)\int\limits_{\xi+\delta}^{x}\frac{1+v\xi}{v-\xi}dv} - \overset{\leftarrow}{\int\limits_{\xi+\delta}^{u}} e^{i\frac{1+v\xi}{v-\xi}T(v)dv} \right) \overset{\rightarrow}{\int\limits_{\xi+\delta}^{u}} e^{-i\frac{1+v\xi}{v-\xi}T(v)dv} \right\| \leq$$
$$\leq \left\| e^{iT(\xi)\int\limits_{\xi+\delta}^{x}\frac{1+v\xi}{v-\xi}dv} - \overset{\leftarrow}{\int\limits_{\xi+\delta}^{u}} e^{i\frac{1+v\xi}{v-\xi}T(v)dv} \right\| \leq \int\limits_{\xi+\delta}^{x} \left\| (T(\xi) - T(v))\frac{1+v\xi}{v-\xi} \right\| dv \leq$$
$$\leq \int\limits_{\xi+\delta}^{x} \frac{\|T(\xi)-T(v)\|}{|v-\xi|}dv + \int\limits_{\xi+\delta}^{x} \frac{|v-\xi|\|\xi T(\xi)\|}{|v-\xi|}dv + \int\limits_{\xi+\delta}^{x} \frac{|\xi|\|\xi T(\xi)-xT(x)\|}{|v-\xi|}dv \leq$$
$$\leq C\int\limits_{\xi+\delta}^{x}(v-\xi)^{\alpha_1-1}dv + C\int\limits_{\xi+\delta}^{x}dv + C|\xi|\int\limits_{\xi+\delta}^{x}(v-\xi)^{\alpha_2-1}dv \leq$$
$$\leq C\int\limits_{\xi}^{x}(v-\xi)^{\alpha_1-1}dv + C\int\limits_{\xi}^{x}dv + C|\xi|\int\limits_{\xi}^{x}(v-\xi)^{\alpha_2-1}dv \leq$$
$$\leq \tfrac{C}{\alpha_1}(x-\xi)^{\alpha_1} + C(x-\xi) + \tfrac{C}{\alpha_2}|\xi|(x-\xi)^{\alpha_2} \leq$$
$$\leq \tfrac{C}{\alpha_1}(x-\xi)^{\alpha} + C(x-\xi)^{\alpha} + \tfrac{C}{\alpha_2}(1+|\xi|)(x-\xi)^{\alpha} \leq \widehat{C}(1+|\xi|)(x-\xi)^{\alpha},$$

i.e.

$$\left\| \left(e^{iT(\xi)\int\limits_{\xi+\delta}^{x}\frac{1+v\xi}{v-\xi}dv} - \overset{\leftarrow}{\int\limits_{\xi+\delta}^{u}} e^{i\frac{1+v\xi}{v-\xi}T(v)dv} \right) \overset{\rightarrow}{\int\limits_{\xi+\delta}^{u}} e^{-i\frac{1+v\xi}{v-\xi}T(v)dv} \right\| \leq \widehat{C}(1+|\xi|)(x-\xi)^{\alpha}. \tag{35}$$

Now the inequalities (31), (32), (33), (34) together with (35) imply that the inequality (29) is true.

The theorem is proved. □

Finally it is worth to mention that using the obtained properties in [2], the presented properties in this paper and similar inequalities and ideas it can be proved the next inequalities concerning the nonnegative or nonpositive matrix function $T(x)$ and the introduced denotations. Let the nonnegative or nonpositive matrix function $T(x)$ be integrable matrix function on $\mathbb{R}$ and satisfying the conditions **(i)** and **(ii)**.

Theorem 5.

$$||\mathcal{U}_{2w}(x) - \mathcal{U}_{2w}(\xi)|| \leq \widetilde{C}(1+|x|)\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} \tag{36}$$

for some constant $\widetilde{C} > 0$*, for all* $w, \xi, x : w < \xi < x$*,* $0 < x - w < 1$ *and* $\alpha' = \alpha/(1+\alpha)$.

Theorem 6.

$$||\mathcal{R}_w^{\pm 1}(\xi) - \mathcal{R}_w^{\pm 1}(x)|| \leq \widetilde{C}(1+|x|)\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} \tag{37}$$

for some constant $\widetilde{C} > 0$*, for all* $w, \xi, x : w < \xi < x$*,* $0 < x - w < 1$ *and* $\alpha' = \alpha/(1+\alpha)$.

Theorem 7.

$$||\mathcal{Q}_w^{\pm}(x) - \mathcal{Q}_w^{\pm}(\xi)|| \leq \widetilde{C}(1+|x|)\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} \tag{38}$$

for some constant $\widetilde{C} > 0$, *for all* $w, \xi, x : w < \xi < x$, $0 < x - w < 1$ *and* $\alpha' = \alpha/(1+\alpha)$.

Theorem 8.

$$||\mathcal{U}_3(x,u) - \mathcal{U}_3(\xi,u)|| \leq \widetilde{C}(1+|x|)\left(\frac{x-\xi}{u-x}\right)^{\alpha'} \tag{39}$$

for $\xi < x < u$, $0 < u - \xi < 1$ *and* $\alpha' = \alpha/(1+\alpha)$.

Theorem 9.

$$||\mathcal{F}_w^{\pm}(\xi,u) - \mathcal{F}_w^{\pm}(x,u)|| \leq \widetilde{C}(1+|x|)\left(\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} + \left(\frac{x-\xi}{u-x}\right)^{\alpha'}\right), \tag{40}$$

$$||\mathcal{U}_w^{\pm}(x,u) - \mathcal{U}_w^{\pm}(\xi,u)|| \leq \widetilde{C}(1+|x|)\left(\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} + \left(\frac{x-\xi}{u-x}\right)^{\alpha'}\right), \tag{41}$$

for some constant $\widetilde{C} > 0$, *for all* $w, \xi, x, u : w < \xi < x < u$, $0 < u - w < 1$, $\alpha' = \alpha/(1+\alpha)$, *where* $U_w(x,u)$ *is defined by the equality*

$$\mathcal{U}_w^{\pm}(x,u) = \mathcal{R}_w^{\mp 1}(x)\mathcal{F}_w^{\pm}(x,u).$$

Theorem 10.

$$\left\| \overset{x}{\underset{w}{\overrightarrow{\int}}} e^{-i\frac{1+v\xi}{v-\xi}T(v)dv} - \mathcal{U}_{2w}(\xi)e^{-iT(\xi)(1+\xi^2)\ln\frac{\xi-x}{\xi-w}} \right\| \leq \widetilde{C}(1+|x|)(\xi-x)^{\alpha'} \tag{42}$$

for some constant $\widetilde{C} > 0$, *for all* $w, \xi, x : w < x < \xi < x + \beta$, $\beta < 1$, *and for each* $\alpha' : 0 < \alpha' \leq \alpha \leq 1$.

These inequalities play an essential role in the process of obtaining the asymptotics of nondissipative curves generated by unbounded operators from the class K^r with different domains of the operator and its adjoint and presented as couplings of dissipative and antidissipative operators with real absolutely continuous spectra.

Finally, it have to mention that all presented inequalities in this paper and in [2] are satisfied when $T(x)$ is nonnegative or nonpositive operator function in infinite dimensional space.

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