BOGOMOLOV MULTIPLIERS FOR SOME p-GROUPS OF NILPOTENCY CLASS 2 WITH 6 GENERATORS

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ABSTRACT: *The Bogomolov multiplier $B_0(G)$ of a finite group G is defined as the subgroup of the Schur multiplier consisting of the cohomology classes vanishing after restriction to all abelian subgroups of G. The triviality of the Bogomolov multiplier is an obstruction to Noether's problem. We show that the Bogomolov multipliers are trivial for six series of 6-generator p-groups of nilpotency class 2.*

KEYWORDS: *Noether's problem, The Bogomolov multiplier.*

1 Introduction

Let K be a field, G a finite group and V a faithful representation of G over K. Then there is a natural action of G upon the field of rational functions $K(V)$. *The rationality problem* (also known as *Noether's problem* when G acts on V by permutations) then asks whether the field of G-invariant functions $K(V)^G$ is rational (i.e., purely transcendental) over K. A question related to the above mentioned is whether $K(V)^G$ is stably rational, that is, whether there exist independent variables $x_1, \dots, x_r$ such that $K(V)^G(x_1, \dots, x_r)$ becomes a purely transcendental extension of K. This problem has close connection with Lüroth's problem [Ša] and the inverse Galois problem [Sa, Sw].

Saltman [Sa] found examples of groups G of order p^9 such that $\mathbb{C}(V)^G$ is not stably rational over $\mathbb{C}$. His main method was application of the unramified cohomology group $H^2_{nr}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ as an obstruction. Bogomolov [Bo] proved that $H^2_{nr}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ is canonically isomorphic to

$$B_0(G) = \bigcap_A \ker\{\mathrm{res}^A_G : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})\}$$

where A runs over all the bicyclic subgroups of G (a group A is called bicyclic if A is either a cyclic group or a direct product of two cyclic groups). The group $B_0(G)$ is a subgroup of the Schur multiplier $H^2(G, \mathbb{Q}/\mathbb{Z})$, and Kunyavskiĭ [Ku] called it the *Bogomolov multiplier* of G. Thus the vanishing of the Bogomolov multiplier is an obstruction to Noether's problem.

Recently, Moravec [Mo1] used a notion of the nonabelian exterior square $G \wedge G$ of a given group G to obtain a new description of $B_0(G)$. Namely, he proved that $B_0(G)$ is (non-canonically) isomorphic to the quotient group $M(G)/M_0(G)$, where $M(G)$ is the kernel of the commutator homomorphism $G \wedge G \to [G, G]$, and $M_0(G)$ is the subgroup of $M(G)$ generated by all $x \wedge y$ such that $x, y \in G$ commute.

Let G be a group and $x, y \in G$. We define $x^y = y^{-1}xy$ and write $[x, y] = x^{-1}x^y = x^{-1}y^{-1}xy$ for the commutator of x and y. We define the commutators of higher weight as $[x_1, x_2, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n]$ for $x_1, x_2, \dots, x_n \in G$.

The nonabelian exterior square of G is a group generated by the symbols $x \wedge y$ ($x, y \in G$), subject to the relations

$$xy \wedge z = (x^y \wedge z^y)(y \wedge z),$$

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$$x \wedge yz = (x \wedge z)(x^z \wedge y^z),$$
$$x \wedge x = 1,$$

for all $x, y, z \in G$. We denote this group by $G \wedge G$. Let $[G, G]$ be the commutator subgroup of G. Obverse that the commutator map $\kappa : G \wedge G \to [G, G]$, given by $x \wedge y \mapsto [x, y]$ is a well-defined group homomorphism. Let $M(G)$ denote the kernel of κ, and $M_0(G)$ detone the subgroup of $M(G)$ generated by all $x \wedge y$ such that $x, y \in G$ commute. Moravec proved in [Mo1] that $B_0(G)$ is (non-canonically) isomorphic to the quotient group $M(G)/M_0(G)$.

There is also an alternative way to obtain the non-abelian exterior square $G \wedge G$. Let φ be an automorphism of G and G^φ be an isomorphic copy of G via $\varphi : x \mapsto x^\varphi$. We define $\tau(G)$ to be the group generated by G and G^φ, subject to the following relations: $[x, y^\varphi]^z = [x^z, (y^z)^\varphi] = [x, y^\varphi]^{z^\varphi}$ and $[x, x^\varphi] = 1$ for all $x, y, z \in G$. Obviously, the groups G and G^φ can be viewed as subgroups of $\tau(G)$. Let $[G, G^\varphi] = \langle [x, y^\varphi] : x, y \in G \rangle$ be the commutator subgroup. Notice that the map $\phi : G \wedge G \to [G, G^\varphi]$ given by $x \wedge y \mapsto [x, y^\varphi]$ is actually an isomorphism of groups (see [BM]).

Now, let $\kappa^* = \kappa \cdot \phi^{-1}$ be the composite map from $[G, G^\varphi]$ to $[G, G]$, $M^*(G) = \ker \kappa^*$ and $M_0^*(G) = \phi(M_0(G))$. Then $B_0(G)$ is clearly isomorphic to $M^*(G)/M_0^*(G)$ by [Mo1]. Notice that

$$M^*(G) = \left\{ \prod_{\text{finite}} [x_i, y_i^\varphi]^{\varepsilon_i} \in [G, G^\varphi] : \varepsilon_i = \pm 1, \prod_{\text{finite}} [x_i, y_i]^{\varepsilon_i} = 1 \right\},$$

and

$$M_0^*(G) = \left\{ \prod_{\text{finite}} [x_i, y_i^\varphi]^{\varepsilon_i} \in [G, G^\varphi] : \varepsilon_i = \pm 1, [x_i, y_i] = 1 \right\}.$$

In order to prove that $B_0(G) = 0$ for a given group G, it suffices to show that $M^*(G) = M_0^*(G)$. This can be achieved by finding a generating set of $M^*(G)$ consisting solely of elements of $M_0^*(G)$.

The advantage of the above description of $G \wedge G$ is the ability of using the full power of the commutator calculus instead of computing with elements of $G \wedge G$. The following two Lemmas collects various properties of $\tau(G)$ and $[G, G^\varphi]$ that will be used in the proofs of our main results.

Lemma 1.1. *([BM]) Let G be a group.*

1. $[x, yz] = [x, z][x, y][x, y, z]$ *and* $[xy, z] = [x, z][x, z, y][y, z]$ *for all* $x, y, z \in G$.
2. *If G is nilpotent of class c, then $\tau(G)$ is nilpotent of class at most $c + 1$.*
3. *If G is nilpotent of class ≤ 2, then $[G, G^\varphi]$ is abelian.*
4. $[x, y^\varphi] = [x^\varphi, y]$ *for all* $x, y \in G$.
5. $[x, y, z^\varphi] = [x, y^\varphi, z] = [x^\varphi, y, z] = [x^\varphi, y^\varphi, z] = [x^\varphi, y, z^\varphi] = [x, y^\varphi, z^\varphi]$ *for all* $x, y, z \in G$.
6. $[[x, y^\varphi], [a, b^\varphi]] = [[x, y], [a, b]^\varphi]$ *for all* $x, y, a, b \in G$.
7. $[x^n, y^\varphi] = [x, y^\varphi]^n = [x, (y^\varphi)^n]$ *for all integers* n *and* $x, y \in G$ *with* $[x, y] = 1$.

8. If $[G,G]$ is nilpotent of class c, then $[G,G^{\varphi}]$ is nilpotent of class c or $c+1$.

Lemma 1.2. *([Mo2, Lemma 3.1]) Let G be a nilpotent group of class ≤ 3. Then*

$$[x,y^n]=[x,y]^n[x,y,y]^{\binom{n}{2}}[x,y,y,y]^{\binom{n}{3}}$$

for all $x,y\in\tau(G)$ and every positive integer n.

2 Bogomolov multipliers for groups of nilpotency class 2

Let p be an odd prime, and let $r\geq 1$ be an integer. Let $H=\langle\beta_1\rangle\times\langle\beta_2\rangle\simeq C_{p^{2r}}\times C_{p^{2r}}$ and $F=\langle\alpha_1\rangle\times\langle\alpha_2\rangle\simeq C_{p^r}\times C_{p^r}$. In this section we are going to show that the Bogomolov multiplier is trivial for six group extensions from $H^2(F,H)$. We list now their presentations:

$$\begin{aligned}
&G_1=\langle\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2:[\beta_1,\alpha_1]=\gamma_1=\beta_2^{p^r},[\beta_2,\alpha_2]=\gamma_2=\beta_1^{p^r}\rangle;\\
&G_2=\langle\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma:[\beta_1,\alpha_1]=[\beta_2,\alpha_2]=\gamma=\beta_2^{p^r}\rangle;\\
&G_3=\langle\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2:[\beta_1,\alpha_1]=\gamma_1=\beta_1^{p^r},[\beta_2,\alpha_2]=\gamma_2=\beta_2^{p^r},[\alpha_1,\alpha_2]=\gamma_2\rangle;\\
&G_4=\langle\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2:[\beta_1,\alpha_1]=\gamma_1=\beta_2^{p^r},[\beta_2,\alpha_2]=\gamma_2=\beta_1^{p^r}\beta_2^{p^r}\rangle;\\
&G_5=\langle\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma:[\beta_1,\alpha_1]=[\beta_2,\alpha_2]=\gamma=\beta_1^{p^r}\beta_2^{p^r}\rangle;\\
&G_6=\langle\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2:[\beta_1,\alpha_1]=\gamma_1=\beta_1^{p^r},[\beta_2,\alpha_2]=\gamma_2=\beta_2^{p^r},[\alpha_1,\alpha_2]=\gamma_1\rangle,
\end{aligned}$$

where all the relations of the form $[x,y]=1$ between the generators have been omitted from the list. Also, according to the assumptions we made in the beginning, we have $\beta_1^{p^{2r}}=\beta_2^{p^{2r}}=\alpha_1^{p^r}=\alpha_2^{p^r}=1$. Thus $\beta_1^{p^r},\beta_2^{p^r}\in Z(G_i)$, i.e., G_i is a p-group of nilpotency class 2 (for $1\leq i\leq 6$).

By examining the relations in these groups we see that they are well defined. It is not hard to see that each G_i is not a direct or a central product of smaller groups. Moreover, each G_i does not have the ABC property.

Theorem 2.1. *If G is isomorphic to any of the groups $G_i, 1\leq i\leq 6$ then $B_0(G)=0$.*

Proof. Case I. $G=G_1$. The group $[G_1,G_1^{\varphi}]$ is generated modulo $M_0^*(G_1)$ by $[\beta_1,\alpha_1^{\varphi}]$ and $[\beta_2,\alpha_2^{\varphi}]$. Since $[G_1,G_1^{\varphi}]$ is abelian, every element $w\in[G_1,G_1^{\varphi}]$ can be written as $w=[\beta_1,\alpha_1^{\varphi}]^m[\beta_2,\alpha_2^{\varphi}]^n\tilde{w}$, where $\tilde{w}\in M_0^*(G_1)$. This gives $w^{\kappa^*}=\gamma_1^m\gamma_2^n$, therefore $w\in M^*(G_1)$ if and only if p^r divides both m and n. By Lemma 1.2 we have that

$$\begin{aligned}
M_0^*(G_1)\ &\ni[\alpha_1^{\varphi},\gamma_2]\\
&=[\alpha_1^{\varphi},\beta_1]^{p^r}[\alpha_1^{\varphi},\beta_1,\beta_1]^{\binom{p^r}{2}}[\alpha_1^{\varphi},\beta_1,\beta_1,\beta_1]^{\binom{p^r}{3}}\\
&=[\alpha_1^{\varphi},\beta_1]^{p^r}[\gamma_1^{-1},\beta_1^{\varphi}]^{\binom{p^r}{2}}[\gamma_1^{-1},\beta_1^{\varphi},\beta_1^{\varphi}]^{\binom{p^r}{3}}\\
&=[\alpha_1^{\varphi},\beta_1]^{p^r},
\end{aligned}$$

and similarly $[\alpha_2^{\varphi},\beta_2]^{p^r}\in M_0^*(G_1)$. Thus we conclude that $w\in M_0^*(G_1)$, hence $B_0(G_1)=0$.

Case II. $G=G_2$. The group $[G_2,G_2^{\varphi}]$ is generated modulo $M_0^*(G_2)$ by $[\beta_1,\alpha_1^{\varphi}]$ and $[\beta_2,\alpha_2^{\varphi}]$. Since $[G_2,G_2^{\varphi}]$ is abelian, every element $w\in[G_2,G_2^{\varphi}]$ can be written as $w=[\beta_1,\alpha_1^{\varphi}]^m[\beta_2,\alpha_2^{\varphi}]^n\tilde{w}$, where $\tilde{w}\in M_0^*(G_2)$. This gives $w^{\kappa^*}=\gamma^{m+n}$, therefore $w\in M^*(G_2)$ if and only if p^r divides $m+n$. By Lemma 1.2 we have that

$$M_0^*(G_2)\ \ni[\alpha_2^{\varphi},\gamma]$$

$$
\begin{aligned}
&= [\alpha_2^\varphi, \beta_2]^{p^r}[\alpha_2^\varphi, \beta_2, \beta_2]^{\binom{p^r}{2}}[\alpha_2^\varphi, \beta_2, \beta_2, \beta_2]^{\binom{p^r}{3}} \\
&= [\alpha_2^\varphi, \beta_2]^{p^r}[\gamma^{-1}, \beta_2^\varphi]^{\binom{p^r}{2}}[\gamma^{-1}, \beta_2^\varphi, \beta_2^\varphi]^{\binom{p^r}{3}} \\
&= [\alpha_2^\varphi, \beta_2]^{p^r}.
\end{aligned}
$$

It follows that

$$M^*(G_2) = \langle [\beta_1, \alpha_1^\varphi][\beta_2, \alpha_2^\varphi]^{-1} \rangle M_0^*(G_2).$$

Note that

$$[\beta_1\alpha_2, \alpha_1\beta_2] = [\beta_1, \beta_2]^{\alpha_2}[\beta_1, \alpha_1]^{\beta_2\alpha_2}[\alpha_2, \beta_2][\alpha_2, \alpha_1]^{\beta_2} = [\alpha_2, \gamma^{-1}] = 1,$$

hence $[\beta_1\alpha_2, (\alpha_1\beta_2)^\varphi] \in M_0^*(G_2)$. Expanding the latter using the class restriction and Lemma 1.1, we get

$$
\begin{aligned}
[\beta_1\alpha_2, (\alpha_1\beta_2)^\varphi] &= [\beta_1, \beta_2^\varphi]^{\alpha_2}[\beta_1, \alpha_1^\varphi]^{\beta_2^\varphi\alpha_2}[\alpha_2, \beta_2^\varphi][\alpha_2, \alpha_1^\varphi]^{\beta_2^\varphi} \\
&= [\beta_1, \beta_2^\varphi]^{\alpha_2}[\beta_1, \alpha_1^\varphi][\beta_1, \alpha_1^\varphi, \beta_2^\varphi\alpha_2][\alpha_2, \beta_2^\varphi][\alpha_2, \alpha_1^\varphi]^{\beta_2^\varphi}.
\end{aligned}
$$

Observe that $[\beta_1, \beta_2^\varphi]^{\alpha_2}, [\beta_1, \alpha_1^\varphi, \beta_2^\varphi\alpha_2]$ and $[\alpha_2, \alpha_1^\varphi]^{\beta_2^\varphi}$ all belong to $M_0^*(G_2)$. Thus we conclude that $[\beta_1, \alpha_1^\varphi][\beta_2, \alpha_2^\varphi]^{-1} = [\beta_1, \alpha_1^\varphi][\alpha_2, \beta_2^\varphi] \in M_0^*(G_2)$, as required.

Case III. $G = G_3$. The group $[G_3, G_3^\varphi]$ is generated modulo $M_0^*(G_3)$ by $[\beta_1, \alpha_1^\varphi], [\beta_2, \alpha_2^\varphi]$ and $[\alpha_1, \alpha_2^\varphi]$. Every element $w \in [G_3, G_3^\varphi]$ can be written as $w = [\beta_1, \alpha_1^\varphi]^m[\beta_2, \alpha_2^\varphi]^n[\alpha_1, \alpha_2^\varphi]^q\tilde{w}$, where $\tilde{w} \in M_0^*(G_3)$. This gives $w^{\kappa^*} = \gamma_1^m\gamma_2^{n+q}$, therefore $w \in M^*(G_3)$ if and only if p^r divides both m and $n+q$. By Lemma 1.2 we have that

$$
\begin{aligned}
1 &= [\alpha_2^\varphi, \alpha_1^{p^r}] \\
&= [\alpha_2^\varphi, \alpha_1]^{p^r}[\alpha_2^\varphi, \alpha_1, \alpha_1]^{\binom{p^r}{2}}[\alpha_2^\varphi, \alpha_1, \alpha_1, \alpha_1]^{\binom{p^r}{3}} \\
&= [\alpha_2^\varphi, \beta_2]^{p^r}[\gamma_2^{-1}, \alpha_1^\varphi]^{\binom{p^r}{2}}[\gamma^{-1}, \alpha_1, \alpha_1^\varphi]^{\binom{p^r}{3}} \\
&= [\alpha_2^\varphi, \alpha_1]^{p^r},
\end{aligned}
$$

and similarly, $[\alpha_1^\varphi, \beta_1]^{p^r}, [\alpha_2^\varphi, \beta_2]^{p^r} \in M_0^*(G_3)$. It follows that

$$M^*(G_3) = \langle [\beta_2, \alpha_2^\varphi][\alpha_1, \alpha_2^\varphi]^{-1} \rangle M_0^*(G_3).$$

Note that

$$[\beta_2\alpha_2, \alpha_2\alpha_1] = [\beta_2, \alpha_1]^{\alpha_2}[\beta_2, \alpha_2]^{\alpha_1\alpha_2}[\alpha_2, \alpha_1][\alpha_2, \alpha_2]^{\alpha_1} = [\alpha_2, \gamma^{-1}] = 1,$$

hence $[\beta_2\alpha_2, (\alpha_2\alpha_1)^\varphi] \in M_0^*(G_3)$. Expanding the latter using the class restriction and Lemma 1.1, we get

$$
\begin{aligned}
[\beta_2\alpha_2, (\alpha_2\alpha_1)^\varphi] &= [\beta_2, \alpha_1^\varphi]^{\alpha_2}[\beta_2, \alpha_2^\varphi]^{\alpha_1^\varphi\alpha_2}[\alpha_2, \alpha_1^\varphi][\alpha_2, \alpha_2^\varphi]^{\alpha_1^\varphi} \\
&= [\beta_2, \alpha_1^\varphi]^{\alpha_2}[\beta_2, \alpha_2^\varphi][\beta_2, \alpha_2^\varphi, \alpha_1^\varphi\alpha_2][\alpha_2, \alpha_1^\varphi][\alpha_2, \alpha_2^\varphi]^{\alpha_1^\varphi}.
\end{aligned}
$$

Observe that $[\beta_2, \alpha_1^\varphi]^{\alpha_2}, [\beta_2, \alpha_2^\varphi, \alpha_1^\varphi\alpha_2]$ and $[\alpha_2, \alpha_2^\varphi]^{\alpha_1^\varphi}$ all belong to $M_0^*(G_3)$. Thus we conclude that $[\beta_2, \alpha_2^\varphi][\alpha_1, \alpha_2^\varphi]^{-1} = [\beta_2, \alpha_2^\varphi][\alpha_2, \alpha_1^\varphi] \in M_0^*(G_3)$, as required.

Case IV. $G = G_4$. This case is similar to Case I. Here $w^{\kappa^*} = \gamma_1^m\gamma_2^n = \beta_1^{np^r}\beta_2^{(m+n)p^r}$, therefore $w \in M^*(G_4)$ if and only if p^r divides both n and $m+n$, i.e., p^r divides both n and m. The proof henceforth is the same as Case I.

Case V. $G = G_5$. This case is similar to Case II. Here $w^{\kappa^*} = \gamma^{m+n} = \beta_1^{(m+n)p^r}\beta_2^{(m+n)p^r}$, therefore $w \in M^*(G_5)$ if and only if p^r divides $m+n$. The proof henceforth is the same as Case II.

Case VI. $G = G_6$. This case is similar to Case III. Here $w^{\kappa^*} = \gamma_1^{m+q}\gamma_2^n$, therefore $w \in M^*(G_6)$ if and only if p^r divides both n and $m+q$. We see that

$$M^*(G_6) = \langle [\beta_1, \alpha_1^{\varphi}][\alpha_1, \alpha_2^{\varphi}]^{-1} \rangle M_0^*(G_6).$$

With similar calculations as in Case III, we can show that $[\beta_1, \alpha_1^{\varphi}][\alpha_1, \alpha_2^{\varphi}]^{-1} \in M_0^*(G_6)$. We are done. □

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