# MÖBIUS TRANSFORMATIONS INDUCED BY ROTATIONS ON THE THREE-SPHERE *

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***ABSTRACT***: *Via the stereographic projection of the unit three-sphere from a pole onto the corresponding equatorial hyperplane it is induced a subgroup of the Möbius group in a three-dimensional Euclidean space. We give a geometric interpretation of any transformations of this group. Applying a quaternion algebra we describe the generated transformations by a quaternionic formalism.*

***KEYWORDS***: *Möbius transformation, stereographic projection, quaternion algebra*

## 1 Introduction

A Möbius transformation in a N-dimensional Euclidean space $\mathbb{E}^N$ on a real vector space $\mathbb{R}^N$ with a standard dot product is the composition of a similarity and an inversion. These mappings are well studied in the case $N = 2$ by the complex formalism $f(z) = \dfrac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$, where $\mathbb{C}$ is the field of the complex numbers, in the complex plane $\mathbb{C}^* = \mathbb{C} \cup \infty$, provided by a point at infinity $\infty$. The Möbius transformations are conformal maps (preserve angles between curves) and under composition of transformations they form a non-commutative group. Many applications in non-Euclidean geometries, an image processing and a geometric modeling use Möbius transformations. In [2] and [3] it is shown an algorithm for constructing a plane curve up to transformation from the subgroup of the Möbius group, induced by rotations on the unit sphere $\mathbb{S}^2 \subset \mathbb{E}^3$, centered at the origin $O$.

In this paper, we consider Möbius transformations in a three-dimensional Euclidean space $\mathbb{E}^3$, generated by rotations on the unit sphere $\mathbb{S}^3$, centered at the origin $O$ and embedded in a four-dimensional Euclidean space $\mathbb{E}^4$. The group of rigid motions on the sphere $\mathbb{S}^3$ coincides with the group $SO(4)$ of rotations in $\mathbb{E}^4$ about the origin. Any element of $SO(4)$ induces a Möbius transformation in $\mathbb{E}^3$ via a stereographic projection . We give an explicit representation of an arbitrary Möbius transformation corresponding to a rotation on the sphere $\mathbb{S}^3$. Furthermore, it is found a quaternion formalism describing the transformations from the obtained subgroup of the Möbius group.

## 2 Preliminaries

Let $\varphi : \mathbb{S}^3 \setminus \{P\} \to \mathbb{E}^3$ be a stereographic projection from a pole $P \in \mathbb{S}^3$. It maps the punctured sphere $\mathbb{S}^3 \setminus \{P\}$ onto the corresponding equatorial hyperplane, identified by a three-dimensional Euclidean space $\mathbb{E}^3$. If we supply $\mathbb{E}^3$ with a point at infinity $\infty$ and correspond it to the pole $P$, i.e. $\varphi(P) = \infty$ then $\varphi : \mathbb{S}^3 \to \mathbb{E}^{3*}$, where $\mathbb{E}^{3*} = \mathbb{E}^3 \cup \infty$, is a bijective map and thus the sphere $\mathbb{S}^3$ is isomorphic to $\mathbb{E}^{3*}$.

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The non-commutative division algebra of quaternions $\mathbb{H}$, invented by William Hamilton, plays an important role in the groups of rotations $SO(4)$ and $SO(3)$ about the origin in $\mathbb{E}^4$ and $\mathbb{E}^3$, respectively. The Hamiltonian algebra of quaternions $\mathbb{H} = \{(a,\ b,\ c,\ d); a, b, c, d \in \mathbb{R}\}$, where $\mathbb{R}$ is the field of real numbers, is a four-dimensional real algebra with base elements $\mathbf{1} = (1, 0, 0, 0)$, $\boldsymbol{i} = (0, 1, 0, 0)$, $\boldsymbol{j} = (0, 0, 1, 0)$, $\boldsymbol{k} = (0, 0, 0, 1)$ and multiplication rules $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$, $\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}$. We can identify $\mathbb{H}$ by the four-dimensional real vector space $\mathbb{R}^4$. The properties of quaternions and their applications are known from [4] and [5]. If $w = (a, b, c, d) \in \mathbb{H}$ is a quaternion, we denote by $\mathcal{S}_w = a \in \mathbb{R}$ the real part of $w$. The quaternions with a zero real part are called pure quaternions. The set of pure quaternions is denoted by $\mathbb{H}_0$ and we have the identifications $\mathbb{H}_0 \cong \mathbb{R}^3$ and $\mathbb{H}_0^* = \mathbb{H}_0 \cup \infty$. Let $\mathcal{V}_w = (b, c, d) \in \mathbb{R}^3$ be the vector corresponding to the pure quaternion $b.\boldsymbol{i} + c.\boldsymbol{j} + d.\boldsymbol{k} \in \mathbb{H}_0$. Then any quaternion $w \in \mathbb{H}$ can be represented uniquely by the sum $w = \mathcal{S}_w + \mathcal{V}_w$ of a real part $\mathcal{S}_w \in \mathbb{R}$ and a vector part $\mathcal{V}_w \in \mathbb{H}_0$. The quaternion $\overline{w} = \mathcal{S}_w - \mathcal{V}_w$ is called a conjugate quaternion of $w$. The norm of quaternion $w = (a, b, c, d)$ is denoted by $\|w\|$ and $\|w\|^2 = a^2 + b^2 + c^2 + d^2 = w.\overline{w}$. If $w \neq 0$ then the inverse quaternion $w^{-1}$ of $w$ is given by $w^{-1} = \dfrac{\overline{w}}{\|w\|^2}$. We can identify the set of unit quaternions $S^3$ by the points on the sphere $\mathbb{S}^3$ and let $P = (1, 0, 0, 0)$. In terms of quaternions, the stereographic projection $\varphi : S^3 \to \mathbb{H}_0^*$ and the reverse stereographic projection $\varphi^{-1} : \mathbb{H}_0^* \to S^3$ have very simple expressions, i.e $\varphi(w) = \dfrac{1 + w}{1 - w} = (1 + w)(1 - w)^{-1}$, $w \in S^3$ and $\varphi^{-1}(q) = \dfrac{q - 1}{1 + q} = (q - 1)(1 + q)^{-1}$, $q \in \mathbb{H}_0^*$. The next two well-known propositions are expressed in terms of quaternions.

**Proposition 1.** *The image of the hyperplane through the origin $O$ with a unit normal vector $a \in S^3$ under the stereographic projection $\varphi : S^3 \to \mathbb{H}_0^*$ is:*

**a)** *a plane through the origin $O$ if and only if $\mathcal{S}_a = 0$;*

**b)** *a sphere with a center $p = -\dfrac{\mathcal{V}_a}{\mathcal{S}_a}$ and a radius $r$, where $r^2 = \|p\|^2 + 1$ if and only if $\mathcal{S}_a \neq 0$.*

**Proposition 2.** *Let $s_a$ be a reflection about a hyperplane in $\mathbb{E}^4$ with a unit normal vector $a \in S^3$ through the origin, $\varphi : S^3 \to \mathbb{H}_0^*$ be the stereographic projection from the sphere $\mathbb{S}^3 \cong S^3$ onto the hyperplane $\mathbb{E}^{3*} \cong \mathbb{H}_0^*$ and let $F = \varphi \circ s_a \circ \varphi^{-1}$ be a transformation in $\mathbb{E}^{3*}$. If $\mathcal{S}_a = 0$ then the transformation $F$ is a reflection about a plane with a normal vector $\boldsymbol{a}$ through the origin $O$. If $\mathcal{S}_a \neq 0$ then $F$ is an inversion $J_p$ with a pole $p = -\dfrac{\mathcal{V}_a}{\mathcal{S}_a} \in \mathbb{H}_0$ and a power $\|p\|^2 + 1$.*

*Proof.* Let $\mathcal{S}_a = 0$. Hence, $a \in \mathbb{H}_0$, $a \in S^3$ and $a^2 = -\|a\|^2 = -1$. Applying the expressions $\varphi^{-1}(q) = \dfrac{q - 1}{1 + q}$, $s_a(w) = -a.\overline{w}.a$ and $\varphi(w) = \dfrac{1 + w}{1 - w}$ of the reverse stereographic projection, the reflection and the stereographic projection, respectively, we obtain that

$$F(q) = \frac{1 - a.\frac{\overline{q}-1}{\overline{q}+1}.a}{1 + a.\frac{\overline{q}-1}{\overline{q}+1}.a} = \frac{-a^2 - a.\frac{\overline{q}-1}{\overline{q}+1}.a}{-a^2 + a.\frac{\overline{q}-1}{\overline{q}+1}.a} = \frac{-a.(1 + \frac{\overline{q}-1}{\overline{q}+1}).a}{-a.(1 - \frac{\overline{q}-1}{\overline{q}+1}).a} = -a.\overline{q}.a = a.q.a,$$

which is a reflection about a plane in $\mathbb{E}^3$ through $O$ with a normal vector $\boldsymbol{a}$ (see [4]).
Let $\mathcal{S}_a \neq 0$. Then

$$(1) \qquad F(q) = \varphi \circ s_a \circ \varphi^{-1}(q) = \frac{2.\mathcal{V}_{aqa} - (\|q\|^2 - 1).\mathcal{V}_{a^2}}{1 + \|q\|^2 + (\|q\|^2 - 1).\mathcal{S}_{a^2} - 2.\mathcal{S}_{aqa}},\ q \in {\mathbb{H}_0}^*.$$

We find that

$$a^2 = (\mathcal{S}_a + \mathcal{V}_a)^2 = \mathcal{S}_a^2 + 2.\mathcal{S}_a.\mathcal{V}_a + \mathcal{V}_a^2 = \mathcal{S}_a^2 - \|\mathcal{V}_a\|^2 + 2.\mathcal{S}_a.\mathcal{V}_a = 2.\mathcal{S}_a^2 - 1 + 2.\mathcal{S}_a.\mathcal{V}_a,$$

$$a.q.a = (\mathcal{S}_a + \mathcal{V}_a).q.(\mathcal{S}_a + \mathcal{V}_a) = \mathcal{S}_a^2.q + \mathcal{S}_a.(\mathcal{V}_a.q + q.\mathcal{V}_a) + \mathcal{V}_a.q.\mathcal{V}_a.$$

Hence,

$$\mathcal{S}_{a^2} = 2.\mathcal{S}_a^2 - 1,\ \mathcal{V}_{a^2} = 2.\mathcal{S}_a.\mathcal{V}_a = -\mathcal{S}_a^2.p,\ \mathcal{S}_{a.q.a} = \mathcal{S}_a.(\mathcal{V}_a.q + q.\mathcal{V}_a) = -\mathcal{S}_a^2.(p.q + q.p),$$
$$\mathcal{V}_{a.q.a} = \mathcal{S}_a^2.(q + p.q.p),$$

where $p = -\dfrac{\mathcal{V}_a}{\mathcal{S}_a}$. Replacing these expressions in (1), after some rearranges and simplifications we get

$$F(q) = (1 + \|p\|^2).\frac{q - p}{\|q - p\|^2} + p,$$

which is the equation of an inversion $J_p$ with a pole $p$ and a power $\|p\|^2 + 1$. □

The four-dimensional rotation group $SO(4)$, that preserves the sphere $\mathbb{S}^3 \subset \mathbb{E}^4$, contains two kinds of rotations: simple rotations and double rotations. In the next two sections we will represent these rotations on the sphere $\mathbb{S}^3 \subset \mathbb{E}^4$ by transformations from the Möbius group in $\mathbb{E}^{3*}$ applying the algebra of quaternions. The complex formalism that represents the rotations on the sphere $\mathbb{S}^2 \subset \mathbb{E}^3$ in the complex plane, provided by a point at infinity, is $f(z) = \dfrac{a.z + b}{-\overline{b}.z + \overline{a}}$, $a, b, z \in \mathbb{C}$, $\|a\|^2 + \|b\|^2 = 1$, where by $\|z\|$ it is denoted the norm of the complex number $z$ and $\overline{z}$ means the complex conjugate of $z$. We will find the quaternion formalism for the Möbius transformations corresponding to the rotations, preserving the sphere $\mathbb{S}^3$.

# 3 Representation of simple rotations on $\mathbb{S}^3$ by Möbius transformations in $\mathbb{H}_0^*$

A simple rotation about a rotation center $O$ on the tree-sphere $\mathbb{S}^3$ leaves a plane $\alpha$ through $O$ a pointwise invariant and induces in the planes $\beta$, that are completely orthogonal to $\alpha$, plane rotations about centers $C_\beta = \alpha \cap \beta$. All these plane rotations have the same rotation angle $\theta$. According to the paper [1], any simple rotation on $\mathbb{S}^3$ through an angle $\theta$ can be represented by the transformation $f(w) = a.w.b$, $a, b, w \in S^3$, where $\mathcal{S}_a = \mathcal{S}_b = \cos\frac{\theta}{2}$. If $s_1(w) = -y.\overline{w}.y$, $s_2(w) = -z.\overline{w}.z$, $y, z, w \in S^3$ are reflections about the hyperplanes in $\mathbb{E}^4 \cong \mathbb{H}$ through the origin, determined by the normal vectors $y$ and $z$, respectively, then $f = s_2 \circ s_1$ and $a.y = y.b = z$.

**Theorem 1.** *Let $f(w) = a.w.b$, $a, b, w \in S^3$, $\mathcal{S}_a = \mathcal{S}_b$ be a simple rotation on the sphere $\mathbb{S}^3$ trough an angle $\theta$, $\varphi : S^3 \to \mathbb{H}_0^*$ be a stereographic projection and let $F = \varphi \circ f \circ \varphi^{-1}$ be the induced transformation in $\mathbb{E}^{3*} \cong \mathbb{H}_0^*$.*
a) *If $b = \overline{a}$ then the transformation $F$ is either the identity transformation $I$ in $\mathbb{H}_0^*$ or a rotation about an axis $\mathbb{R}_{\mathcal{V}_a}$ through the angle $\theta$, where $\cos\frac{\theta}{2} = \mathcal{S}_a$ and $\mathbb{R}_{\mathcal{V}_a}$ is the one-dimensional vector space, defined by the nonzero vector $\mathcal{V}_a$;*
b) *If $b \neq \overline{a}$ then the transformation $F$ is either the composition $F = J_l \circ J_p$ of inversions $J_l$ and $J_p$ with poles $p = \dfrac{\mathcal{V}_{ab}}{1 - \mathcal{S}_{ab}}$ and $l = \dfrac{\mathcal{V}_{a^2b} - \mathcal{V}_a}{\mathcal{S}_a - \mathcal{S}_{a^2b}}$, respectively, provided that $\mathcal{S}_a = \mathcal{S}_b \neq 0$, or the composition $F = G_z \circ J_p$, where $G_z$ is a reflection about a plane, provided that $\mathcal{S}_a = \mathcal{S}_b = 0$.*

*Proof.* Let $b = \overline{a}$ and $\mathcal{V}_a = 0$. Then $a = b = 1$ and therefore $F \equiv I$. If $\mathcal{V}_a \neq 0$ then any unit pure quaternion $y \in \mathbb{H}_0$, orthogonal to $\mathcal{V}_a$, satisfies the condition $a.y = y.\overline{a} = z \in \mathbb{H}_0$. Hence, $f(w) = z.\overline{y}.w.\overline{y}.z = s_z \circ s_y(w)$, where $s_y$ and $s_z$ are reflections about the hyperplanes, defined by $y$ and $z$, respectively. From Proposition 2 we find that $F = \varphi \circ s_z \circ s_y \circ \varphi^{-1} = \varphi \circ s_z \circ \varphi^{-1} \circ \varphi \circ s_y \circ \varphi^{-1} = G_z \circ G_y$, where $G_z$ and $G_y$ are reflections about planes in $\mathbb{E}^3$. Therefore, $F(q) = G_z \circ G_y(q) = (z.y).q.(y.z) = a.q.\overline{a}$ is a rotation in $\mathbb{E}^3$ about an axis $\mathbb{R}_{\mathcal{V}_a}$ through the angle $\theta$, where $\cos\frac{\theta}{2} = \mathcal{S}_a$ (see [4]).
Let $b \neq \overline{a}$ and $\mathcal{S}_a = \mathcal{S}_b \neq 0$. Hence, $\mathcal{V}_b \neq -\mathcal{V}_a \Leftrightarrow$

$$(2) \qquad < \mathcal{V}_b, \mathcal{V}_b > \neq - < \mathcal{V}_a, \mathcal{V}_b > \Leftrightarrow 1 - \mathcal{S}_b^2 \neq - < \mathcal{V}_a, \mathcal{V}_b > \Leftrightarrow \mathcal{S}_a^2 - < \mathcal{V}_a, \mathcal{V}_b > -1 \neq 0,$$

where by $< \cdot, \cdot >$ is denoted the standard dot product in $\mathbb{E}^3$. From Lemma 2.2 in [1] it follows that there exists a unit quaternion $y = \dfrac{a.b - 1}{\|a.b - 1\|}$ such that $a.y = y.b = z$ and $f = s_z \circ s_y$, where $s_z$ and $s_y$ are reflections about hyperplanes. Using (2) we obtain that

$$\|a.b - 1\|.\mathcal{S}_y = \mathcal{S}_{ab-1} = \mathcal{S}_{ab} - 1 = \mathcal{S}_a^2 - < \mathcal{V}_a, \mathcal{V}_b > -1 \neq 0 \Rightarrow \mathcal{S}_y \neq 0 \ \text{ and}$$

$$\|a.b - 1\|.\mathcal{S}_z = \mathcal{S}_{a^2b-a} = \mathcal{S}_{a^2b} - \mathcal{S}_a = \mathcal{S}_{a^2}\mathcal{S}_b - < \mathcal{V}_{a^2}, \mathcal{V}_b > -\mathcal{S}_a =$$
$$= (2\mathcal{S}_a^2 - 1)\mathcal{S}_a - 2\mathcal{S}_a < \mathcal{V}_a, \mathcal{V}_b > -\mathcal{S}_a = 2\mathcal{S}_a.(\mathcal{S}_a^2 - < \mathcal{V}_a, \mathcal{V}_b > -1) \neq 0 \Rightarrow \mathcal{S}_z \neq 0.$$

Thus, from Proposition 2 it follows that $F = J_l \circ J_p$, where $p = -\dfrac{\mathcal{V}_y}{\mathcal{S}_y} = \dfrac{\mathcal{V}_{ab}}{1 - \mathcal{S}_{ab}}$ and $l = -\dfrac{\mathcal{V}_z}{\mathcal{S}_z} = \dfrac{\mathcal{V}_{a^2b} - \mathcal{V}_a}{\mathcal{S}_a - \mathcal{S}_{a^2b}}$.
If $\mathcal{S}_a = \mathcal{S}_b = 0 \Leftrightarrow a^2 = b^2 = -1$ then $\mathcal{S}_y \neq 0$, but $\|a.b - 1\|.\mathcal{S}_z = \mathcal{S}_{a^2b-a} = -\mathcal{S}_b - \mathcal{S}_a = 0 \Rightarrow \mathcal{S}_z = 0$ and $F = G_z \circ J_p$, where $G_z$ is a reflection about a plane, defined by $z \in \mathbb{H}_0$. □

**Corollary 1.** *Let $f(w) = c.w.c$, $c \in S^3$, $\mathcal{S}_c \neq 0$, $\mathcal{V}_c \neq 0$ be a simple rotation on the sphere $\mathbb{S}^3$, $\varphi : S^3 \to \mathbb{H}_0^*$ be a stereographic projection and $F = \varphi \circ f \circ \varphi^{-1}$ be the induced transformation in $\mathbb{E}^{3*} \cong \mathbb{H}_0^*$. Then $F$ is the compositions $F = J_l \circ J_p = J_0 \circ J_{p''} = J_{p'} \circ J_0$ of two inversions, where*

$$(3) \qquad p = \frac{\mathcal{S}_c}{1 - \mathcal{S}_c^2}.\mathcal{V}_c, \ \ l = \frac{2.\mathcal{S}_c^2 - 1}{2.\mathcal{S}_c^2}.p$$

*and*

$$(4) \qquad p' = -p'' = -\frac{\mathcal{V}_c}{\mathcal{S}_c}.$$

*Proof.* The equalities $F = J_l \circ J_p$ and (3) follow immediately from the Theorem 1b), when $a = b = c \neq \overline{c}$ (since $c \notin \mathbb{R}$).
Another way to represent the rotation $f$ is the product $f = f_2 \circ f_1$ of reflections $f_1(w) = -\overline{w}$ and $f_2(w) = -c.\overline{w}.c$. Also, $f = f_1 \circ f_2'$, where $f_2'$ is the reflection $f_2'(w) = -\overline{c}.\overline{w}.\overline{c}$. Hence, according to Proposition 2, we obtain that $F$ is the compositions $F = J_0 \circ J_{p''} = J_{p'} \circ J_0$ of inversions $J_0$ after $J_{p''}$ and $J_{p'}$ after $J_0$, where $p'$ and $p''$ are determined by (4). □

# 4 Representation of double rotations on the sphere $\mathbb{S}^3$ by Möbius transformations in $\mathbb{E}^{3^*}$

The double rotations in $\mathbb{E}^4$ about a center $O$, preserving the sphere $\mathbb{S}^3$, leave only the point $O$ invariant. Any double rotation has at least one pair of completely orthogonal 2-planes $\alpha$ and $\beta$ through $O$, that are invariant, but they aren't a pointwise invariant. In general, the rotation angle $\theta$ in $\alpha$ and the rotation angle $\psi$ in $\beta$ are different.

First, we will consider the transformation $f(w) = c^2.w$ on the sphere $\mathbb{S}^3 \subset \mathbb{E}^4$, where $c, w \in S^3$, $\mathcal{S}_c \neq 0$, $\mathcal{V}_c \neq 0$. It is clear that $f = f_2 \circ f_1 = f_1 \circ f_2$, where $f_1(w) = c.w.c$ and $f_2 = c.w.\bar{c}$. According to Theorem 1a) and Corollary 1 we have that the transformation $F = \varphi \circ f \circ \varphi^{-1}$, induced by the stereographic projection $\varphi : S^3 \to \mathbb{H}_0^*$ in $\mathbb{H}_0^*$, is $F = R_{\mathcal{V}_c}^{\theta} \circ J_l \circ J_p = J_l \circ J_p \circ R_{\mathcal{V}_c}^{\theta}$, where $R_{\mathcal{V}_c}^{\theta}$ is a rotation about an axis $\mathbb{R}_{\mathcal{V}_c}$ through the angle $\theta$, $\cos\frac{\theta}{2} = \mathcal{S}_c$. The poles $l$ and $p$ of inversions $J_l$ and $J_p$, respectively, are defined by (3).

Similarly, the transformation $f(w) = w.c^2$ on the unit sphere $\mathbb{S}^3 \subset \mathbb{E}^4$ is a commute product of rotations $f_1(w) = c.w.c$ and $f_2(w) = \bar{c}.w.c$. Then the corresponding transformation $F = \varphi \circ f \circ \varphi^{-1}$, induced by the stereographic projection $\varphi : S^3 \to \mathbb{H}_0^*$ in $\mathbb{H}_0^*$, is $F = R_{\mathcal{V}_c}^{-\theta} \circ J_l \circ J_p = J_l \circ J_p \circ R_{\mathcal{V}_c}^{-\theta}$, where the axis of rotation $R_{\mathcal{V}_c}^{-\theta}$, the angle $\theta$, the poles $l$ and $p$ of the inversions $J_l$ and $J_p$, respectively, are defined as above.

**Theorem 2.** *Let $f(w) = a.w$, $a \in S^3$, $\mathcal{V}_a \neq 0$ be a rotation on the sphere $\mathbb{S}^3$, $\varphi : S^3 \to \mathbb{H}_0^*$ be a stereographic projection and $F = \varphi \circ f \circ \varphi^{-1}$ be the induced transformation in $\mathbb{E}^{3^*} \cong \mathbb{H}_0^*$. Then $F$ is the compositions $F = R_{\mathcal{V}_a}^{\theta} \circ J_l \circ J_p = R_{\mathcal{V}_a}^{\theta} \circ J_0 \circ J_{p''} = R_{\mathcal{V}_a}^{\theta} \circ J_{p'} \circ J_0$ or $F = J_l \circ J_p \circ R_{\mathcal{V}_a}^{\theta} = J_0 \circ J_{p''} \circ R_{\mathcal{V}_a}^{\theta} = J_{p'} \circ J_0 \circ R_{\mathcal{V}_a}^{\theta}$ of a rotation and two inversions, where*

$$(5)\qquad p = \frac{1}{1-\mathcal{S}_a}.\mathcal{V}_a, \quad l = \frac{\mathcal{S}_a}{1+\mathcal{S}_a}.p, \quad \cos\theta = \mathcal{S}_a$$

*and*

$$(6)\qquad p' = -p'' = -\frac{\mathcal{V}_a}{1+\mathcal{S}_a}.$$

*Proof.* We set $a = c^2$, $c \in S^3$, $\mathcal{S}_c \neq 0$, $\mathcal{V}_c \neq 0$ and we find that $\mathcal{S}_c^2 = \dfrac{1+\mathcal{S}_a}{2}$, $\mathcal{V}_c = \dfrac{\mathcal{V}_a}{2.\mathcal{S}_c}$. Applying the last relations and using the equalities (3) and (4) we get (5) and (6). □

Analogously, we have the next theorem:

**Theorem 3.** *Let $f(w) = w.b$, $b \in S^3$, $\mathcal{V}_b \neq 0$ be a rotation on the sphere $\mathbb{S}^3$, $\varphi : S^3 \to \mathbb{H}_0^*$ be a stereographic projection and $F = \varphi \circ f \circ \varphi^{-1}$ be the induced transformation in $\mathbb{E}^{3^*} \cong \mathbb{H}_0^*$. Then $F$ is the compositions $F = R_{\mathcal{V}_a}^{-\psi} \circ J_m \circ J_n = R_{\mathcal{V}_b}^{-\psi} \circ J_0 \circ J_{n''} = R_{\mathcal{V}_b}^{-\psi} \circ J_{n'} \circ J_0$ or $F = J_m \circ J_n \circ R_{\mathcal{V}_b}^{-\psi} = J_0 \circ J_{n''} \circ R_{\mathcal{V}_b}^{-\psi} = J_{n'} \circ J_0 \circ R_{\mathcal{V}_b}^{-\psi}$ of a rotation and two inversions, where*

$$(7)\qquad n = \frac{1}{1-\mathcal{S}_b}.\mathcal{V}_b, \quad m = \frac{\mathcal{S}_b}{1+\mathcal{S}_b}.n, \quad \cos\psi = \mathcal{S}_b$$

*and*

$$(8)\qquad n' = -n'' = -\frac{\mathcal{V}_b}{1+\mathcal{S}_b}.$$

The transformations $f(w) = a.w$, $a \in S^3$ and $f(w) = w.b$, $b \in S^3$ in $\mathbb{E}^4 \cong \mathbb{H}$ are so called left and right transformation, respectively. Clearly, left(right) transformations form a group and every left transformation commutes with every right transformation. Also, these transformations are known as Clifford translations. Obviously, any double rotation is a product of a left and a right transformation. Hence, summarizing Theorem 2 and Theorem 3 we obtain the next theorem.

**Theorem 4.** *Let $f(w) = a.w.b$, $a, b \in S^3$, $\mathcal{S}_a \neq \mathcal{S}_b$ be a double rotation on the sphere $\mathbb{S}^3$, $\varphi : S^3 \to \mathbb{H}_0^*$ be a stereographic projection and $F = \varphi \circ f \circ \varphi^{-1}$ be the induced transformation in $\mathbb{E}^{3*} \cong \mathbb{H}_0^*$. Then $F$ is the compositions $F = F_1 \circ F_2 = F_2 \circ F_1$, where $F_1 = R^{\theta}_{\mathcal{V}_a} \circ J_l \circ J_p = R^{\theta}_{\mathcal{V}_a} \circ J_0 \circ J_{p''} = R^{\theta}_{\mathcal{V}_a} \circ J_{p'} \circ J_0$ or $F_1 = J_l \circ J_p \circ R^{\theta}_{\mathcal{V}_a} = J_0 \circ J_{p''} \circ R^{\theta}_{\mathcal{V}_a} = J_{p'} \circ J_0 \circ R^{\theta}_{\mathcal{V}_a}$, $F_2 = R^{-\psi}_{\mathcal{V}_a} \circ J_m \circ J_n = R^{-\psi}_{\mathcal{V}_b} \circ J_0 \circ J_{n''} = R^{-\psi}_{\mathcal{V}_b} \circ J_{n'} \circ J_0$ or $F_2 = J_m \circ J_n \circ R^{-\psi}_{\mathcal{V}_b} = J_0 \circ J_{n''} \circ R^{-\psi}_{\mathcal{V}_b} = J_{n'} \circ J_0 \circ R^{-\psi}_{\mathcal{V}_b}$ and $p$, $l$, $\theta$, $m$, $n$, $\psi$ are defined by the equalities (5), (6), (7) and (8).*

The transformations $F$, generated by the rotations $f \in SO(4)$ on the unit sphere $\mathbb{S}^3$ via the stereographic projection $\varphi$, form a six-parameter subgroup of the Möbius group Möb(3) in $\mathbb{E}^3$. By the next theorem we give a quaternion formalism for an arbitrary element of this subgroup.

Möbius transformations in a four-dimensional Euclidean space $\mathbb{E}^4$, supplied by a point at infinity $\infty$, are well determined by the quaternion formalism $f(w) = (a.w + b).(c.w + d)^{-1}$, $a, b, c, d \in \mathbb{H}$ (see [6]), where the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is assumed to be invertible and the determinant $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \|a.d - a.c.a^{-1}.b\|^2 = 1$. The product of two Möbius transformations $f_2$ after $f_1$ with matrixes $M_2$ and $M_1$, respectively, is a Möbius transformation $f = f_2 \circ f_1$ with a matrix $M = M_2.M_1$ that is the matrix product of $M_2$ and $M_1$.

**Theorem 5.** *Any Möbius transformation in $\mathbb{E}^{3*}$, corresponding to a rotation on the sphere $\mathbb{S}^3 \subset \mathbb{E}^4$, can by represented by the quaternion formalism*

$$F(q) = (A.q + B).(B.q + A)^{-1},\ q \in \mathbb{H}_0,\ A, B \in \mathbb{H}, \tag{9}$$

*where $\|A\|^2 + \|B\|^2 = 1$ and $A.\overline{B} \in \mathbb{H}_0$.*

*Proof.* Let $f(w) = a.w.b = a.w.(b^{-1})^{-1} = a.w.\bar{b}^{\,-1}$, $a, b, w \in S^3$ be a rotation on the sphere $\mathbb{S}^3$. Then the corresponding Möbius transformation $F$ in $\mathbb{E}^{3*}$ via the stereographic projection $\varphi$ is the transformation $F = \varphi \circ f \circ \varphi^{-1}$ with a matrix

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \begin{pmatrix} a & 0 \\ 0 & \bar{b} \end{pmatrix} . \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} . \begin{pmatrix} a + \bar{b} & \bar{b} - a \\ \bar{b} - a & a + \bar{b} \end{pmatrix}.$$

Denoting $A = \dfrac{a + \bar{b}}{2}$ and $B = \dfrac{\bar{b} - a}{2}$, we obtain (9). Moreover, it is easy to find that $\|A\|^2 + \|B\|^2 = 1$ and $A.\overline{B} \in \mathbb{H}_0$.

Conversely, let $F$ be a Möbius transformation in $\mathbb{E}^{3*}$, defined by (9). Then the corresponding transformation $f$ on the sphere $\mathbb{S}^3$ via the stereographic projection $\varphi$ is $f = \varphi^{-1} \circ F \circ \varphi$. That is a Möbius transformation with a matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \begin{pmatrix} A & B \\ B & A \end{pmatrix} . \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} A - B & 0 \\ 0 & A + B \end{pmatrix}.$$

Since $\|A\|^2+\|B\|^2=1$ and $A.\overline{B}+B.\overline{A}=0$ then $\|A-B\|^2=\|A\|^2+\|B\|^2-A.\overline{B}-B.\overline{A}=1$ and $\|A+B\|^2=\|A\|^2+\|B\|^2+A.\overline{B}+B.\overline{A}=1$. Hence $f(w)=a.w.\overline{b}^{-1}=a.w.b$, where $a=A-B\in S^3$, $b=\overline{A+B}\in S^3$, is a rotation on the sphere $\mathbb{S}^3$. □

**Example 1.** *Let* $c(t)=\{t.\sin(3.t),\, t.\cos(3.t),\, t\}$, $t\in\mathbb{R}$ *be a conic spiral and f(w)=a.w.b, where*
$a=\cos\alpha+\dfrac{\sin\alpha}{\sqrt{3}}.(\mathbf{i}+\mathbf{j}+\mathbf{k})$, $b=\cos\alpha+\dfrac{\sin\alpha}{\sqrt{2}}.(\mathbf{i}+\mathbf{k})$, *be a single rotation on the sphere* $\mathbb{S}^3$. *One-parameter set of Möbius curves equivalent to* $c=c(t)$ *under Möbius transformations, induced by* $f$ *in* $\mathbb{E}^3$ *via a stereographic projection* $\varphi$, *is represented on Figure 1.*

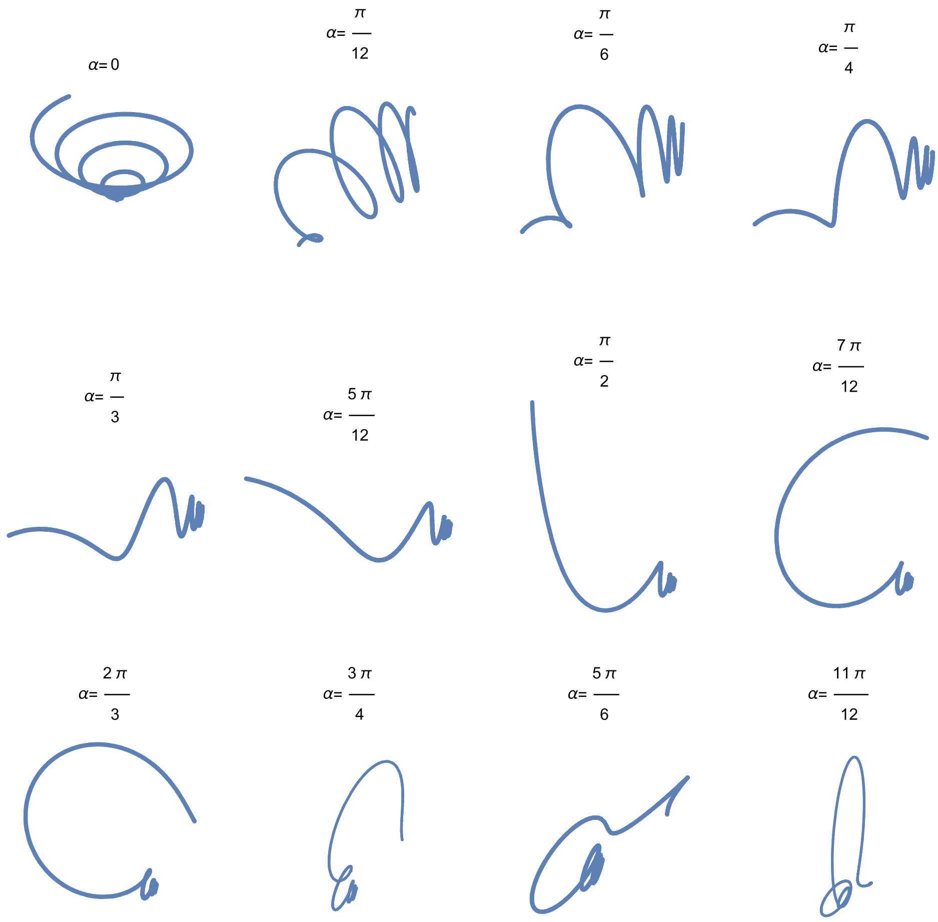


Figure 1: One-parameter set of Möbius curves

**Example 2.** *Let* $c(t)=\{t.\sin(3.t),\, t.\cos(3.t),\, t\}$, $t\in\mathbb{R}$ *be a conic spiral and f(w)=a.w.b, where*
$a=\dfrac{1}{\sqrt{3}}+\sqrt{\dfrac{2}{3}}(\cos\alpha.\mathbf{i}+\sin\alpha.\mathbf{k})$, $b=\dfrac{1}{2}+\dfrac{1}{2}.(\mathbf{i}+\mathbf{j}+\mathbf{k})$, *be a double rotation on the sphere* $\mathbb{S}^3$. *One-parameter set of Möbius curves equivalent to* $c=c(t)$ *under Möbius transformations, induced by* $f$ *in* $\mathbb{E}^3$ *via a stereographic projection* $\varphi$, *is represented on Figure 2.*

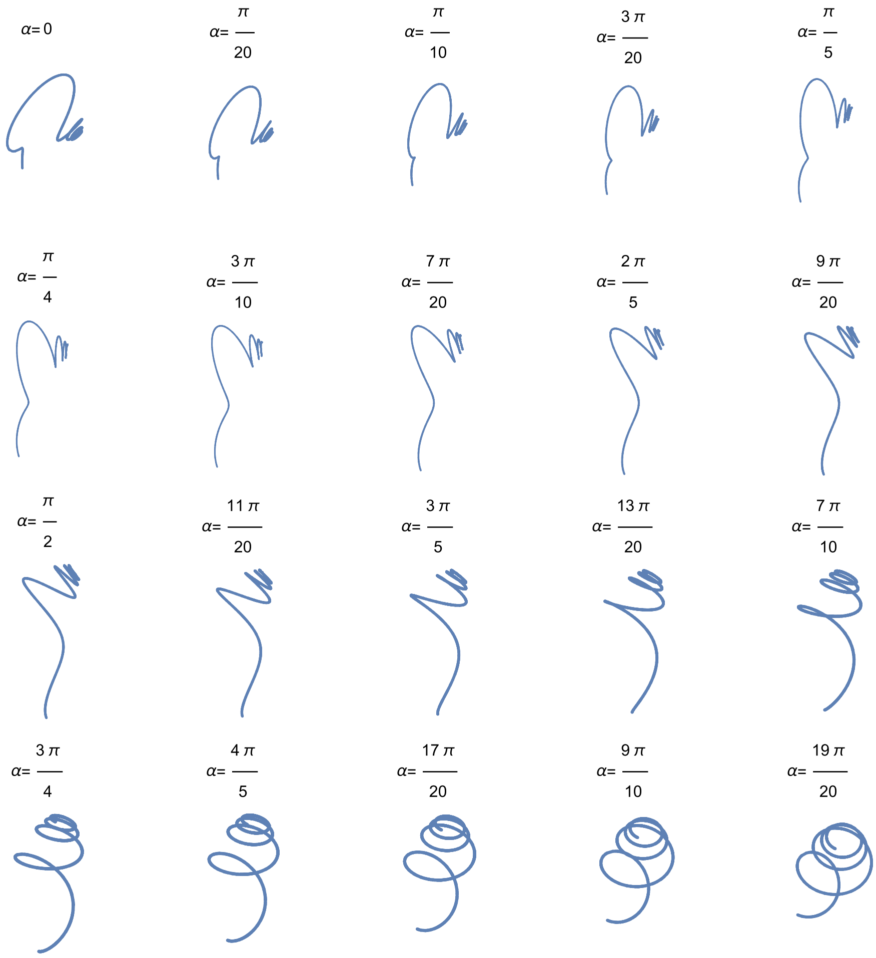


Figure 2: One-parameter set of Möbius curves

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