ON THE PHASE PORTRAIT OF A *p*-LAPLACIAN DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

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***ABSTRACT*:** *We consider the differential equation* $\left(|u'|^{p-2}u'\right)' - a|u|^{q-2}u + b|u|^{r-2}u = 0$, *where* $a>0$, $b>0$ *are given constants;* $p>1$, $q>1$ *and* $r>1$ *are real difference numbers and study it's phase portraits.*

***KEYWORDS*:** *Phase portraits of a p-Laplacian differential equations, Conservation Law.*

1. Introduction

For $p>1$ let us define the function $\varphi_p(t) = |t|^{p-2}t$.
We are interested on the phase portraits of the following differential equation

$$\left(|u'|^{p-2}u'\right)' - a|u|^{q-2}u + b|u|^{r-2}u = 0 \qquad \textbf{(1)}$$

where $a>0$, $b>0$ are given constants; $p>1$, $q>1$ and $r>1$ are real difference numbers. We will consider the behavior of phase curves depending on the numbers p, q, r and using MAPLE program will get phase portraits, describing the solution in the given equation. The equations of this tips appear in biomathematical and phase transition models, see [1] and [3]. When $p = 2$, the equation (1) is considered in Chaparova, J., Kalcheva [2] and [4]. They made an analysis of the phase plane in various cases $q > 1$, $r > 1$ and phase portraits are presented.
The differential equation **(1)** can be rewritten as the system

$$\begin{cases} u' = \varphi_{p'}(v) \\ v' = a\varphi_q(u) - b\varphi_r(u) \end{cases} \qquad \textbf{(2)}$$

from which it follows the identity

$$\phi_{p'}(v) - a\phi_q(u) + b\phi_r(u) = C ,$$

known as is Conservation law.
Further we'll use the function

$$V(u) = -a\frac{|u|^q}{q} + b\frac{|u|^r}{r}$$

for $a>0$, $b>0$ and $p>1$, $q>1$ and $r>1$ and plot the phase portraits of the system **(2)**.

2. Main results and phase portraits

In the present paper we deal with the following differential equation

$$\left(|u'|^{p-2}u'\right)' - a|u|^{q-2}u + b|u|^{r-2}u = 0,$$

where $a>0$, $b>0$ are given constants; $p>1$, $q>1$ and $r>1$ are real difference numbers.

We introduce the functions

$$\varphi_p(t) = |t|^{p-2}t, \quad \phi_p(t) = \frac{|t|^p}{p} \text{ where } p>1 \text{ and } \phi'_p(t) = \varphi_p(t).$$

The equation **(1)** can be rewritten as $\left(\varphi_p\left(u'\right)\right)' - a\varphi_q\left(u\right) + b\varphi_r\left(u\right) = 0$.

We have

Proposition 1: If $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ then $\varphi_{p'} = \varphi_p^{-1}$ (3)**.**

Proof: We have

$$\varphi_{p'}\left(\varphi_p\left(t\right)\right) = \left|\left|t\right|^{p-2} t\right|^{p'-2} \left|t\right|^{p-2} t = \left(\left|t\right|^{p-1}\right)^{\frac{p}{p-1}-2} \left|t\right|^{p-2} t = \left|t\right|^{p-2p+2+p-2} t = t$$

which proves **(3).**

Let $v = \left|u'\right|^{p-2} u' \;\Rightarrow\; u' = \varphi_{p'}\left(v\right)$ and $v' = a\varphi_q\left(u\right) - b\varphi_r\left(u\right)$**.**

The equation **(1)** can be rewritten as the system **(2)**
from

$$\varphi_{p'}\left(v\right).v' - a.\varphi_q\left(u\right).u' + b.\varphi_r\left(u\right).u' = 0 \quad \Rightarrow$$

$$\Rightarrow \quad \left(\phi_{p'}\left(v\right) - a\phi_q\left(u\right) + b\phi_r\left(u\right)\right)'_t = 0$$

or

$$\phi_{p'}\left(v\right) - a\phi_q\left(u\right) + b\phi_r\left(u\right) = C \; ,$$

known as is **Conservation law.**
The stationary points of the system **(2)** are the solutions of the system

$$\left|\begin{array}{l} v = 0 \\ u\left|u\right|^{q-2}\left(a - b\left|u\right|^{r-q}\right) = 0 \end{array}\right.$$

and are the points

$$\left(0,0\right); \left(\left(\frac{a}{b}\right)^{\frac{1}{r-q}}, 0\right); \left(-\left(\frac{a}{b}\right)^{\frac{1}{r-q}}, 0\right).$$

Further, we consider the behavior of phase curves depending on the numbers p, q and r**.**
We use the program MAPLE to plot the phase portraits, describing the solution in the given equation. We'll use the function $V\left(u\right) = -a\dfrac{\left|u\right|^q}{q} + b\dfrac{\left|u\right|^r}{r}$**.**

<u>Case 1</u>. Let $0 < q < r$**.**

Since $V'\left(u\right) = -au\left|u\right|^{q-2} + bu\left|u\right|^{r-2} = u\left|u\right|^{q-2}\left(-a + b\left|u\right|^{r-q}\right)$

$$V'\left(u\right) = 0 \Leftrightarrow u_1 = 0, u_{2,3} = \pm\left(\frac{a}{b}\right)^{\frac{1}{(r-q)}}.$$

For $0 < q < r$ and $u \in \left(-\left(\dfrac{a}{b}\right)^{\frac{1}{q-r}}, 0\right) \cup \left(\left(\dfrac{a}{b}\right)^{\frac{1}{q-r}}, +\infty\right)$ $V\left(u\right)$ is increasing function;

For $0<q<r$ and $u\in\left(-\infty,-\left(\frac{a}{b}\right)^{\frac{1}{q-r}},\right)\cup\left(0,\left(\frac{a}{b}\right)^{\frac{1}{q-r}}\right)$ $V(u)$ is decreasing function. We have

$$V_{\max}(u_1=0)=0 \text{ – local maximum,}$$

$$V_{\min}\left(u_2=\pm\left(\frac{a}{b}\right)^{\frac{1}{q-r}}\right)=\left(\frac{a}{b}\right)^{\frac{q}{r-q}}\left(\frac{-ar+aq}{qr}\right)\text{– global minimums.}$$

Let's describe the phase portraits.

Let *q*=1.5, *r*=3, *a*=2, *b*=3. The stationary point $(u_1,0)$ corresponds to the energy $h_0=V_{\max}(u_0=0)=0$ and the stationary points $(u_2,0)$ and $(u_3,0)$ to the energy $h_2=V_{\min}\left(u=\pm\left(\frac{a}{b}\right)^{\frac{1}{q-r}}\right)=-0.44$. The energy level $h_0=0$ generates two phase curves which are symmetric – so called separatrisies of the saddle. The energy level $h_1=-0.2$ as $h_2<h_1<h_0$ is generated two closed curves contained inside the points - two centers $(u_2,0)$ and $(u_3,0)$.

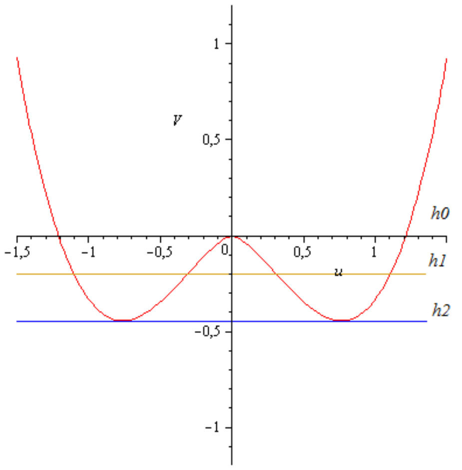


Figure 1 Graph of the function $V(u)$ in *q*=1.5, *r*=3, *a*=2, *b*=3.

Since

$$\phi_{p'}(v)-a\phi_q(u)+b\phi_r(u)=C\Rightarrow\phi_{p'}(v)+V(u)=C\Rightarrow$$
$$V(u)=C-\phi_{p'}(v)$$

Below we shall plot the graph of $V(u)=C-\phi_{p'}(v)$.

$$B:=implicitplot\left(V(u)=-0.4-\phi_{p'}(v),u=0..1.5,v=0..1.2,color=green\right):$$
$$C:=implicitplot\left(V(u)=-0.3-\phi_{p'}(v),u=0..1.5,v=0..1.2,color=yellow\right):$$
$$E:=implicitplot\left(V(u)=-0.1-\phi_{p'}(v),u=0..1.5,v=0..1.2,color=blue\right):$$
$$F:=implicitplot\left(V(u)=-\phi_{p'}(v),u=0..1.5,v=0..1.2,color=orange\right):$$
$$H:=implicitplot\left(V(u)=0.2-\phi_{p'}(v),u=0..1.5,v=0..1.2,color=red\right):$$

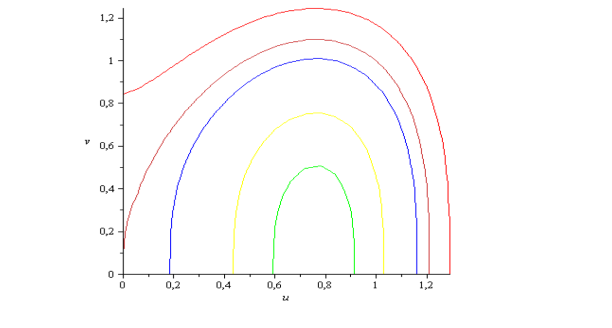
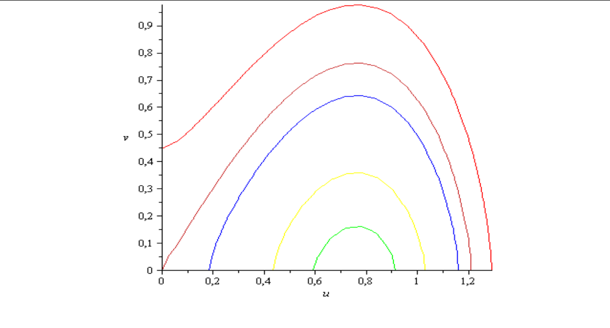
	
$p=1.5,\ p'=3, q=1.5,\ r=3,\ a=2,\ b=3$	$p=3,\ p'=1.5, q=1.5,\ r=3,\ a=2,\ b=3$

Figure 2 Graph of $V(u)=C-\phi_{p'}(v)$

On **Figure 3** is plotted the graph of the function $f(u,v)=\phi_{p'}(v)-a\phi_q(u)+b\phi_r(u)$ for $p=1.5,\ p'=3,\ q=1.5,\ r=3,\ a=2,\ b=3.$

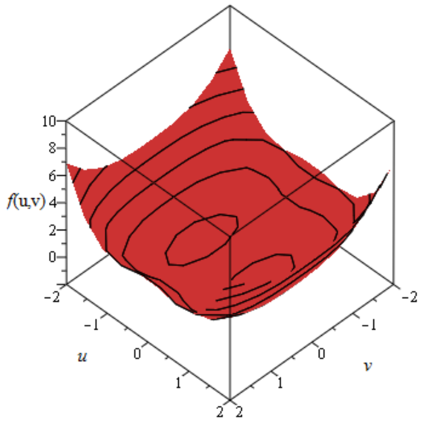


Figure 3 Graph of the function $f(u,v)=\phi_{p'}(v)-a\phi_q(u)+b\phi_r(u)$.

The phase portraits are plotted on **Figure 4 and Figure 5** with Computer algebra system **MAPLE** for $p=1.5,\ p'=3,\ q=1.5,\ r=3,\ a=2,\ b=3$ and $p=3,\ p'=1.5,\ q=1.5,\ r=3,\ a=2,\ b=3.$

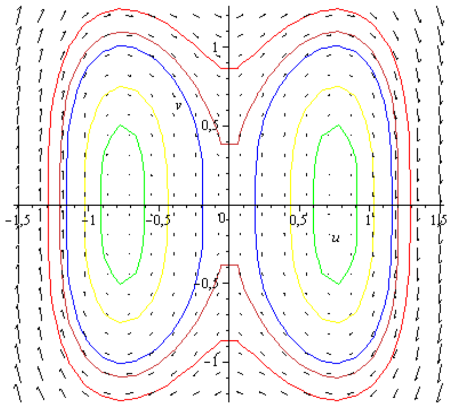
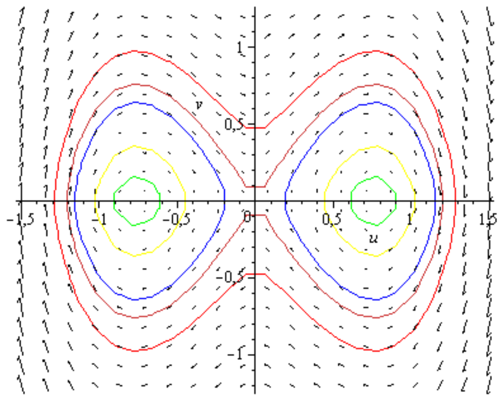
	
$p=1.5,\ p'=3, q=1.5,\ r=3,\ a=2,\ b=3$	$p=3,\ p'=1.5, q=1.5,\ r=3,\ a=2,\ b=3$

Figure 4 Phase portrait of $\phi_{p'}(v)-a\phi_q(u)+b\phi_r(u)=C$.

The phase portrait on **Figure 4** shows two groups of closed curves, generated by the potential wells (with a minimum of $V(u)$), which correspond to the periodic solutions; two homoclinics going on from the beginning and entering the same stationary point and at another closed curve obtained by positive energy value C=**0.2**.

One can obtain some properties of stationary solutions.

Proposition 2. Let $\frac{a}{b}>1$, $1<q<r$. Then, if in addition $r-q\to\infty$ then $u_{2,3}\to\pm1$; if $r-q\to0$ then $u_{2,3}\to\pm\infty$.

Proof:_We have

$$\lim_{(r-q)\to\infty}u_{2,3}=\lim_{(r-q)\to\infty}\pm\left(\frac{a}{b}\right)^{\frac{1}{r-q}}=\pm1\ \text{ if }\ \frac{a}{b}>1,$$

$$\lim_{(r-q)\to0}u_{2,3}=\lim_{(r-q)\to0}\pm\left(\frac{a}{b}\right)^{\frac{1}{r-q}}=\pm\infty\ \text{ if }\ \frac{a}{b}>1.$$

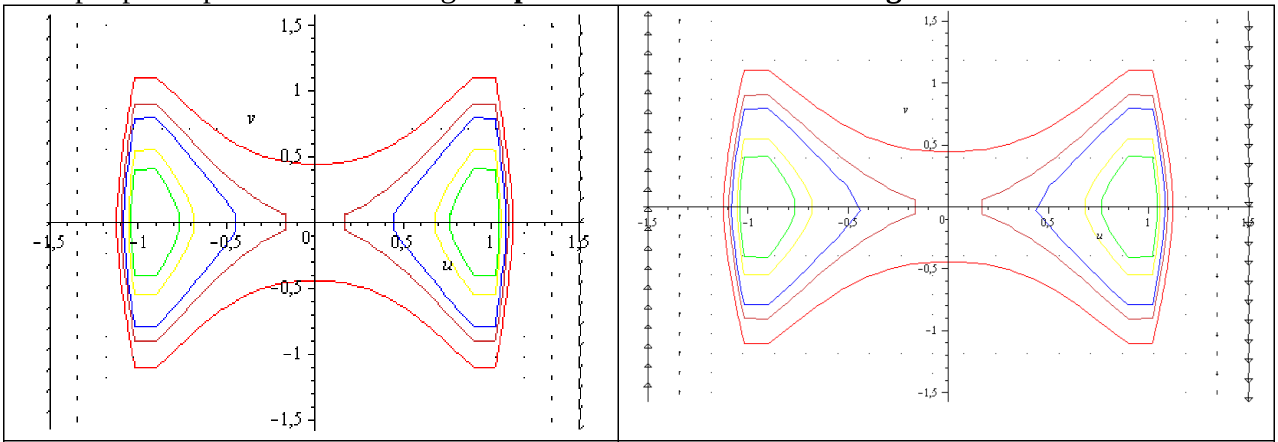
Proposition 3. Let $0<\frac{a}{b}<1$, $1<q<r$. Then, if in addition $r-q\to\infty$ then $u_{2,3}\to\pm1$; if $r-q\to0$ then $u_{2,3}\to0$.

Proof:_We have

$$\lim_{(r-q)\to\infty}u_{2,3}=\lim_{(r-q)\to\infty}\pm\left(\frac{a}{b}\right)^{\frac{1}{r-q}}=\pm1\ \text{ if }\ 0<\frac{a}{b}<1,$$

$$\lim_{(r-q)\to0}u_{2,3}=\lim_{(r-q)\to0}\pm\left(\frac{a}{b}\right)^{\frac{1}{r-q}}=0\ \text{ if }\ 0<\frac{a}{b}<1$$

Sample phase portraits illustrating **Proposition 1.2** are shown on **Figure 5.**



$p=1.5$, q=2.5, r=15, a=2, b=3, p'=3	$p=1.5$, q=2.5, r=15, a=2, b=3, p'=3

Figure 5 Phase portrait of $\phi_{p'}(v)-a\phi_q(u)+b\phi_r(u)=C$

Case 2:_Let $1<r<q$.

Since

$$V'(u)=-au|u|^{q-2}+bu|u|^{r-2}=u|u|^{r-2}\left(-a|u|^{q-r}+b\right)\ \Rightarrow$$

the function $V(u)$ for $u \in \left(-\infty, -\left(\frac{a}{b}\right)^{\frac{1}{q-r}}\right) \cup \left(0, \left(\frac{a}{b}\right)^{\frac{1}{q-r}}\right)$ is increasing and for

$u \in \left(-\left(\frac{a}{b}\right)^{\frac{1}{q-r}}, 0\right) \cup \left(\left(\frac{a}{b}\right)^{\frac{1}{q-r}}, +\infty\right)$ is decreasing.

At $u_1 = 0$ the function $V(u)$ has a local minimum and $V_{\min}(u_1 = 0) = 0,$

and at $u_{2,3} = \pm\left(\frac{a}{b}\right)^{\frac{1}{q-r}}$ – global maximums

$$V_{\max}\left(u_{2,3} = \pm\left(\frac{a}{b}\right)^{\frac{1}{r-q}}\right) = \left(\frac{a}{b}\right)^{\frac{r}{q-r}}\left(\frac{-a^2 r + b^2 q}{bqr}\right).$$

Let's describe the phase portraits for $q=3,\ r=\frac{3}{2},\ a=2,\ b=3$. Let the stationary point $(u_1, 0)$ corresponds to the energy $h_0 = V_{\min}(u_1 = 0) = 0$ and the stationary points $(u_2, 0)$ and $(u_3, 0)$ to the energy $h_2 = V_{\max} = 1.5$ as $h_0 < h_1 < h_2 < h_3$. Energy h_1 is generated a closed phase curve and other two symmetrical curves forming saddles in the points $(u_2, 0)$ and $(u_3, 0)$. Energy h_2 are generated two symmetrical phase curves, connecting the stationary points $(u_2, 0)$ and $(u_3, 0)$. These curves correspond to the heteroclinic solutions: when $t \to -\infty$ to approach to to first stationary point and when $t \to +\infty$ to approach to other stationary point. Energies h_1, h_2 are generated also more two curves. We plot the graph of the function $V(u)$.

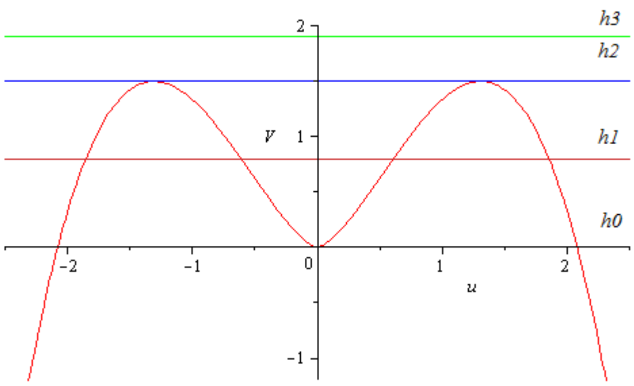


Figure 6. Graph of the function $V(u)$ for $q=3,\ r=\frac{3}{2},\ a=2,\ b=3.$

On **Figure 7** are plotted the graph of the $V(u) = C - \phi_{p'}(v)$ for $p' = 3, q=3, r=\frac{3}{2}, a=2, b=3$ and $p' = 1.5, q=3, r=\frac{3}{2}, a=2, b=3.$

$$B := implicitplot\left(V(u) = -0.7 - \frac{|v|^{p'}}{p'}, u = 0..3, v = 0..5, color = green\right)$$

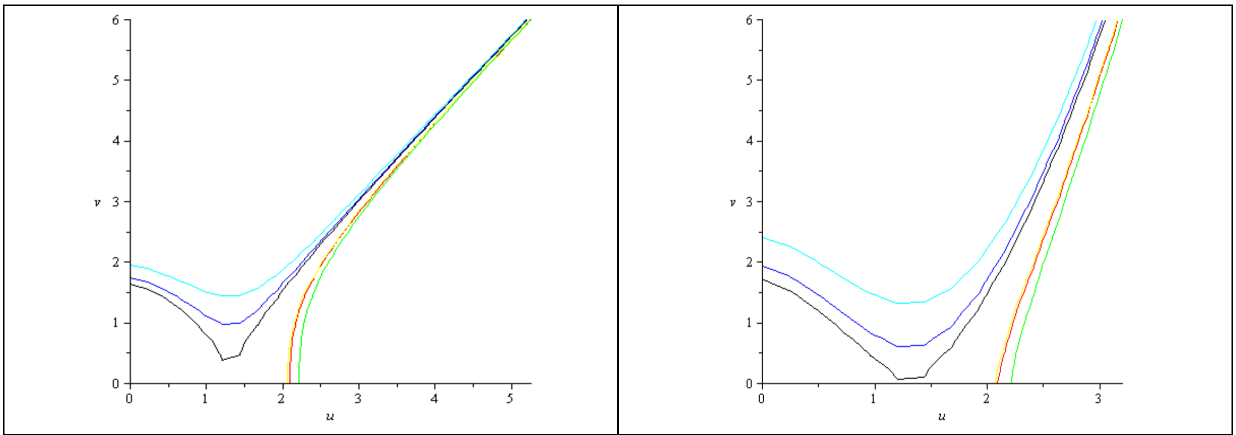
$$E := implicitplot\left(V(u) = -\frac{|v|^{p'}}{p'}, u = 0..3, v = 0..5, color = yellow\right)$$

$$C := implicitplot\left(V(u) = -0.1 - \frac{|v|^{p'}}{p'}, u = 0..3, v = 0..5, color = red\right)$$

$$F := implicitplot\left(V(u) = 1.8 - \frac{|v|^{p'}}{p'}, u = 0..3, v = 0..5, color = blue\right)$$

$$H := implicitplot\left(V(u) = 1.5 - \frac{|v|^{p'}}{p'} u = 0..3, v = 0..5, color = black\right)$$

$$G := implicitplot\left(V(u) = 2.5 - \frac{|v|^{p'}}{p'}, u = 0..3, v = 0..5, color = cyan\right)$$



$p' = 3, q=3,\ r=\frac{3}{2},\ a=2,\ b=3.$	$p' = 1.5, q=3,\ r=\frac{3}{2},\ a=2,\ b=3.$

Figure 7 Graph of $V(u) = C - \phi_{p'}(v)$

And on **Figure 8** is shown the graph of the function $f(u,v) = \phi_{p'}(v) - a\phi_q(u) + b\phi_r(u)$ for $p' = 3, q=3,\ r=\frac{3}{2},\ a=2,\ b=3$.

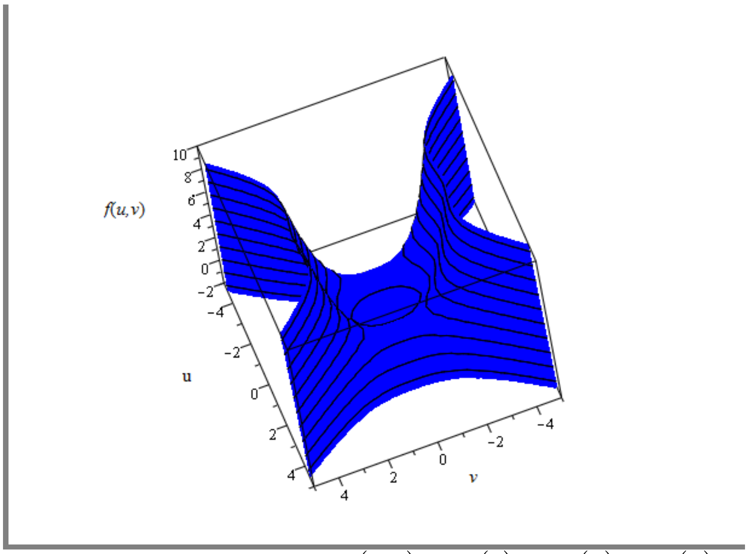
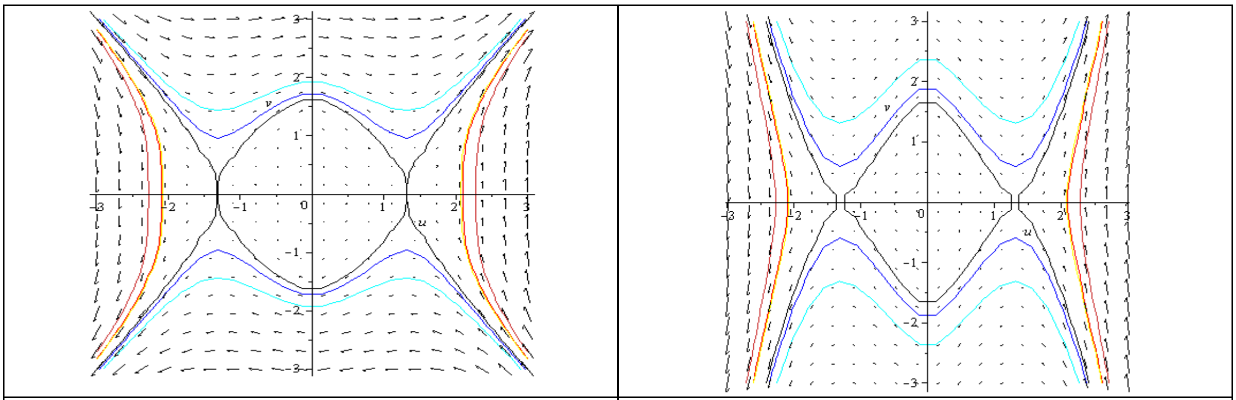


Figure 8. Graph of the function $f(u,v) = \phi_{p'}(v) - a\phi_q(u) + b\phi_r(u)$.

On **Figure 9** are plotted the Phase Portraits of $\phi_{p'}(v) - a\phi_q(u) + b\phi_r(u) = C$ for $p = \frac{3}{2},\ q{=}3,\ r{=}\frac{3}{2},\ a{=}2,\ b{=}3,\ s{=}3$ and $p = 3,\ q{=}3,\ r{=}\frac{3}{2},\ a{=}2,\ b{=}3,\ s{=}1.5$.



$p = \frac{3}{2},\ q{=}3,\ r{=}\frac{3}{2},\ a{=}2,\ b{=}3,\ s{=}3$	$p = 3,\ q{=}3,\ r{=}\frac{3}{2},\ a{=}2,\ b{=}3,\ s{=}1.5$

Figure 9 Phase portrait of $\phi_{p'}(v) - a\phi_q(u) + b\phi_r(u) = C$

The Phase portrait is observed as a set of closed curves around the local minimum $V(u)$; two heteroclinics and two more curves obtained at energies ***C*=1.8** and ***C*=2.5**.
As before we obtain the following properties.

Proposition 4. Let $\frac{a}{b} > 1$, $1 < r < q$. If in addition $r - q \to -\infty$ then $u_{2,3} \to \pm 1$; if $r - q \to 0$ then $u_{2,3} \to 0$.

Proof: We have

$$\lim_{(r-q)\to -\infty} u_{2,3} = \lim_{(r-q)\to -\infty} \pm \left(\frac{a}{b}\right)^{\frac{1}{r-q}} = \pm 1 \text{ if } \frac{a}{b} > 1,$$

$$\lim_{(r-q)\to 0} u_{2,3} = \lim_{(r-q)\to 0} \pm \left(\frac{a}{b}\right)^{\frac{1}{r-q}} = 0 \text{ if } \frac{a}{b} > 1.$$

Proposition 5. Let $0 < \frac{a}{b} < 1$, $1 < r < q$. If in addition $r - q \to -\infty$ then $u_{2,3} \to \pm 1$; if $r - q \to 0$ then $u_{2,3} \to \pm\infty$.

Acknowledgments

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REFERENCES

[1] Grossinho, M., Minhós, F., Tersian, S. *Positive Homoclinic Solutions for a Class of Second Order Differential Equations*. // Journal of Mathematical Analysis and Applications, 1999, № 240, p. 163-173.
[2] Chaparova, J., Kalcheva, E. *Existence and multiplicity of periodic solutions of second-order ODE with sublinear and superlinear terms*. // Proceedings of the Union of Scientists-Ruse, 2014, №11, p. 14-22.
[3] Chaparova, J., Peletier, L., Tersian, S. *Existence and nonexistence of nontrivial solutions of semilinear fourth and sixth-order differential equations*. // Advances in Differential equations, 2003, vol. 8 , book 10, p. 1237-1258.
[4] Kalcheva, E. *Multiplicity of periodic solutions of semilinear second-order ODE. Dissertation*, 2014, p.45-56.