АЛГЕБРИ НА БУРГЕН НА ПОДАЛГЕБРИ НА $A(\bar{D})$ ВЪРХУ ЕДИНИЧНИЯ КРЪГ

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BOURGAIN ALGEBRAS OF SUBALGEBRAS OF $A(\bar{D})$ ON THE UNIT DISK

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ABSTRACT: *Let* ψ *be a finite Blaschke product and* $A(\bar{D})$ - denote the disk algebra. *In this paper we prove that the Bourgain algebra of* $\psi A(\bar{D})$ *relative to* $L^\infty(D)$ *is contained in* $(H^\infty(D)\cap W(D))+C(\bar{D})+V$.

KEYWORDS: *Disk algebra; Bounded analytic functions; Bourgain algebra.*

1. Introduction

Let Y be a commutative Banach algebra with an identity, and X be a linear subspace of Y . J. Cima and R. Timoney introduced the notion of the Bourgain algebra $([1])$. The Bourgain algebra X_b or $(X,Y)_b$ of X relative to Y is the space of all functions f in Y such that $\operatorname{dist}(f.f_n, X)\to 0$ for every sequence $\{f_n\}_n$ converging weakly to zero in X (i.e. such that $\varphi(f_n)\to 0$ for every bounded linear functional φ on X). The distance between $f.f_n$ and X is the quotient norm of the coset $f.f_n+X$ in the space Y/X . J. Cima and R.Timoney proved in $[1]$ that X_b is a closed subalgebra of Y and if X is an algebra then $X\subset X_b$.

Let D be the open unit disk and let $T=\{z:|z|=1\}$ be its boundary – the unit circle. Let $H^\infty(D)$ be the algebra of bounded analytic functions in D. Taking the boundary values of the functions on T we can consider $H^\infty(D)$ as a closed subalgebra $H^\infty(T)$ of $L^\infty(T)$- the algebra of essentially bounded measurable functions with respect to the Lebesque measure $d\theta/2\pi$ on T. Let $C(\bar{D})$ be the space of all continuous functions on the closed unit disk $\bar{D}$ and let A denote the disk algebra, i.e. the algebra of all continuous functions on $\bar{D}$ which are analytic on D. There are various alternative descriptions of the disk algebra. For example: $A=A(\bar{D})$ is the uniform closure in $C(\bar{D})$ of the polynomials, also $A=A(T)$ consists of the continuouse functions on the unit circle whose Fourier coefficients vanish on the negative integers.

There are various different natural spaces Y containing $H^\infty(D)$. In $[2]$ J. Cima, K. Stroethoff and K. Yale characterized the Bourgain algebra $(H^\infty(D), L^\infty(D))_b$ of $H^\infty(D)$ with respect to algebra $L^\infty(D)$ of Lebesgue measurable, essentially bounded functions on the unit dick D. They showed that

$$\left(H^{\infty}(D),L^{\infty}(D)\right)_b=H^{\infty}(D)+C(\bar{D})+V,$$

where $C(\bar{D})$ is the algebra of continuous functions on closed unit disk $\bar{D}$ and

$$V=\left\{g\in L^{\infty}(D):\|g.\chi_{D\backslash rD}\|_{\infty}\to 0 \text{ as } r\to 1^{-}\right\}$$

is the ideal of functions in $L^{\infty}(D)$ which vanish in an appropriate sense near the boundary T.

For $f\in L^{\infty}(D)$, $\xi\in T$ and $t>0$ we put

$$\omega(f,\xi,t)=ess\sup\left\{|f(z)-f(w)|:z,w\in E(\xi,t)\right\},$$

where $E(\xi,t)=\left\{z\in D:|z-\xi|<t\right\}$. Essential oscillation of f at the point $\xi\in T$ is defined as

$$\omega(f,\xi)=\lim_{t\to 0^{+}}\omega(f,\xi,t).$$

Let $W(D)$ be the set of functions in $L^{\infty}(D)$ whose essential oscillations are bounded away from zero at only finitely many points of the unit circle, i.e.

$$W(D)=\left\{f\in L^{\infty}(D):\text{for every }\delta>0\text{ the set }\left\{\xi\in T:\omega(f,\xi)\geq\delta\right\}\text{ is finite}\right\}.$$

For $f\in L^{\infty}(T)$ and $\xi\in T$ the essential oscillation $\omega(f,\xi)$ is defined analogously and we define

$$W=\left\{f\in L^{\infty}(T):\text{for every }\delta>0\text{ the set }\left\{\xi\in T:\omega(f,\xi)\geq\delta\right\}\text{ is finite}\right\}.$$

In $[3]$ K. Izuchi proved that $\left(A(T),L^{\infty}(T)\right)_b=\left(H^{\infty}(T)\cap W\right)+C(T)$. In $[4]$ J. Cima, K. Stroethoff and K. Yale have proved that

$$\left(A(\bar{D}),L^{\infty}(D)\right)_b=\left(H^{\infty}(D)\cap W(D)\right)+C(\bar{D})+V.$$

Let ψ be a finite Blaschke product, i.e. ψ is a function in $H^{\infty}(D)$ of the type

$$\psi(z)=z^{p}\prod_{n=1}^{k}\left[\frac{-\bar{z}_n}{|z_n|}\frac{z-z_n}{1-\bar{z}_n z}\right]^{p_n}, \text{ where:}$$

1) $p_1,p_2,...,p_k$ are non-negative integers;

2) $z_1,z_2,...,z_k$ are different non-zero numbers from the open unit disk.

In [5], [6] and [7] are described the Bourgain algebras of certain subalgebras on $H^{\infty}(D)$. In [8] it has proven that $\left(\psi A(\bar{D}),L^{\infty}(D)\right)_b\supset\left(H^{\infty}(D)\cap W(D)\right)+C(\bar{D})+V$ in the case when ψ is a finite Blaschke product. In this paper we prove the opposite inclusion.

2. Preliminaries.

A sequence $\{z_n\}_n$ in D is called interpolating if for every bounded sequence $\{a_n\}_n$ of complex numbers there is a function $f\in H^{\infty}(D)$ such that $f(z_n)=a_n$ for all n. For a sequence $\{z_n\}_n$ in D with $\sum_{n=1}^{\infty}(1-|z_n|)<\infty$, the function:

$$B(z)=\prod_{n=1}^{\infty}\frac{-\bar{z}_n}{|z_n|}\frac{z-z_n}{1-\bar{z}_n z},\ z\in D,$$

is called a Blaschke product with zeros $\{z_n\}_n$. If $\{z_n\}_n$ is an interpolating sequence, then $B(z)$ is also called interpolating. The study of interpolating sequences is useful in many areas of function theory and operator theory. Some results for interpolating sequences and their applications in the description of Douglas algebras have been obtained in $[9],[10]$ and $[11]$. Interpolating sequences can be applied for obtaining new scalar solutions of nonlinear differential equations using results in $[12]$ and $[13]$.

There are two important statements for the interpolating sequences.

Lemma 2.1. $([14])$. If $\{z_n\}_n \subset D$ is interpolating sequence, then there exist functions $\{f_n\}_n \subset H^\infty(D)$ and positive number M such that $f_n(z_n)=1$ for all n, $f_n(z_k)=0$ for $n \neq k$ and $\sum_{n=1}^{\infty}|f_n(z)| \leq M$ for $z \in D$.

Lemma 2.2. $([14])$. Suppose that $\{f_n\}_n$ is a sequence in $H^\infty(D)$ such that $\sum_{n=1}^{\infty}|f_n(z)| \leq M$ for $z \in D$. Then $f_n \to 0$ weakly in $H^\infty(D)$.

The following lemma is used to provide the weakly null sequence obtained formerly from lemma 2.1 and lemma 2.2.

Lemma 2.3. Let $\{z_n\}_n$ be a sequence in D such that $|z_n| \to 1$ as $n \to \infty$, and $\delta > 0$. Then there exists a subsequence $\{z_{n_j}\}_j$ of $\{z_n\}_n$ and a sequence $\{f_j\}_j$ in $\psi A(\bar{D})$ such that $f_j \to 0$ weakly in $\psi A(\bar{D})$, $\|f_j\| \leq 4\delta$ and $|f_j(z_{n_j})| \geq \delta$ for all j .

Proof: By considering a subsequence of $\{z_n\}_n$, we may assume that $z_n \to 1$. Let $g(z) = \frac{z+1}{2}e^{-i\theta_0} \in A(\bar{D})$, where $e^{-i\theta_0} \in \psi(1)$. Then $F = \psi.g \in \psi A(\bar{D})$, $F(1) = \psi(1)g(1) = 1$ and $|F(z)| = |\psi(z)|.|g(z)| < 1$ for every $z \in \bar{D} \setminus \{1\}$. Let $\{s_j\}_j$ be a sequence of positive integers such that $s_j \to \infty$. Since $F^{s_j}(1) = 1$, we can choose a subsequence $\{z_{n_j}\}_j$ of $\{z_n\}_n$ such that $|F^{s_j}(z_{n_j})| > \frac{1}{2}$.

Fix j. Since $|F(z_{n_j})| < 1$ then $|1 - F^t(z_{n_j})| \to 1$ as $t \to \infty$. Hence there exists $t = t_j$ such that $|1 - F^{t_j}(z_{n_j})| \geq \frac{1}{2|F^{s_j}(z_{n_j})|}$, becausse $\frac{1}{2|F^{s_j}(z_{n_j})|} < 1$. In this way we obtain a sequence $t_j \to \infty$ such that

$$\left|F^{s_j}(z_{n_j})\left(1 - F^{t_j}(z_{n_j})\right)\right| \geq \frac{1}{2}.$$

Put $h_j = 2.F^{s_j}\left(1 - F^{t_j}\right)$ for every j. Then h_j belongs to $\psi A(\bar{D}) \subset C(\bar{D})$, becausse $2F^{s_j} \in \psi A(\bar{D})$ and $1 - F^{t_j} \in \psi A(\bar{D})$; $h_j(1) = 0$ and

$$\left\|h_j\right\| = \max_{\bar{D}}\left|h_j(z)\right| \le \max_{\bar{D}} 2\left|F(z)\right|^{s_j}\left(1+\left|F(z)\right|^{t_j}\right) \le 4 ,$$

i.e. the sequence $\{h_j\}_j$ is uniformly bounded. On the other hand

$$h_j(z) = 2.F^{s_j}(z)\left(1-F^{t_j}(z)\right) \to 0$$

for every $z \in \bar{D}$, because $F(1)=1$ and $\left|F(z)\right|<1$ for $z \in \bar{D}\setminus\{1\}$. Therefore $h_j \to 0$ weakly in $C(\bar{D})$ and $\phi(h_j) \to 0$ for every $\phi \in \left(C(\bar{D})\right)^*$. Let $\varphi \in \left(\psi A(\bar{D})\right)^*$ and $\phi \in \left(C(\bar{D})\right)^*$ is the Hahn Banach extension of φ on $C(\bar{D})$. Then $\varphi(h_j) = \phi(h_j) \to 0$ and $h_j \to 0$ weakly in $\psi A(\bar{D})$.

If $\delta > 0$ then the sequence $\{f_j\}_j$, where $f_j = \delta.h_j$ for every j, belongs to $\psi A(\bar{D})$, $f_j \to 0$ weakly in $\psi A(\bar{D})$ and $\left\|f_j\right\| \le 4\delta$. Also we have

$$\left|f_j\left(z_{n_j}\right)\right| = \delta\left|h_j\left(z_{n_j}\right)\right| = \delta\left|2F^{s_j}\left(z_{n_j}\right)\left(1-F^{t_j}\left(z_{n_j}\right)\right)\right| \ge \delta$$

for all j.

□

This lemma, in the case of $A(\bar{D})$ has been proved in $[15]$.

3. The main result

For $\xi \in T$ and $R \in (0,1)$ the open non-tangential cone $\Gamma_R(\xi)$ at ξ is the interior of the convex hull of ξ and the disk $\{z \in \mathbb{C} : |z| \le R\}$. We say $f \in L^\infty(D)$ that has essential non-tangential limit at ξ if

$$ess\sup\left\{\left|f(z)-L\right| : z \in \Gamma_R(\xi), |z| > 1-\delta\right\} \to 0 \text{ as } \delta \to 0^+$$

for all $R \in (0,1)$. In this case we will write $f^*(\xi)$ for L. Define BV to be the set of all $f \in L^\infty(D)$ such that an essential non-tangential limit $f^*(\xi)$ exists for almost every $\xi \in T$.

Let $g \in L^\infty(T)$ and

$$\tilde{g}(z) = g\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t).g\left(e^{i\theta}\right)dt \ , \ z \in D, \ 0 \le r < 1$$

is the extension of g to D via the Poisson kernel. Then $\tilde{g}$ is a bounded harmonic function in D (in particular $\tilde{g} \in L^\infty(D)$) , $g\left(e^{i\theta}\right) = \lim_{r\to 1}\tilde{g}\left(r.e^{i\theta}\right)$ almost everywhere on the unit circle T and $\|\tilde{g}\|_D = \|g\|_T$.

Theorem 3.1. Let ψ be a finite Blaschke product. If $f \in \left(\psi A(\bar{D}), L^\infty(D)\right)_b$, then $f^* \in \left(\psi A(T), L^\infty(T)\right)_b$ and $f^* \in W(D)$.

Proof: If $f \in \left(\psi A(\bar{D}), L^\infty(D)\right)_b$, then by ($[16]$, theorem 2.2) the non-tangential limit $f^*(\zeta)$ exists at almost every point of $\zeta \in T$ and $f^* \in L^\infty(T)$. Let the sequence $\{g_n\}_n$ be weakly null in $\psi A(T) = \psi A(\bar{D})$. Because $f \in \left(\psi A(\bar{D}), L^\infty(D)\right)_b$ there exists a sequence

$\{p_n\}_n \subset \psi A(\bar{D})$ such that $\| fg_n - p_n\|_\infty \to 0$. Since the mapping $F \to F^*$ is a contractive homomorphism of BV in $L^\infty(T)$ ([2]), we have $\| f^* g_n^* - p_n^*\|_T \le \| fg_n - p_n\|_\infty \to 0$, where $g_n^* = g_n \in \psi A(T)$ and $p_n^* = p_n \in \psi A(T)$. Therefore $f^* \in (\psi A(T), L^\infty(T))_b$. Since the Shilov boundary of $\psi A(\bar{D})$ is the unit circle T and each point in T is a peak point for $\psi A(\bar{D})$, then by [3] we get that $f^* \in W$.

Let there exists $\delta > 0$ such that the set $\left\{\xi \in T : \omega\left(f^*, \xi\right) \ge \delta\right\}$ is infinity. By ([4], lemma 5)

$$\omega\left(f^*, \xi\right) \le \omega\left(f^*, \xi\right) \le 2\omega\left(f^*, \xi\right)$$

for every $\xi \in T$ and we obtain that the set $\left\{\xi \in T : \omega\left(f^*, \xi\right) \ge \frac{\delta}{2}\right\}$ is also infinity. This contradicts to $f^* \in W$. Consequently for every $\delta > 0$ the set $\left\{\xi \in T : \omega\left(f^*, \xi\right) \ge \delta\right\}$ is finity and $f^* \in W(D)$.

□

Let E be a Lebesgue measurable subset of $\mathbb{R}^k$ and we denote by $|E|$ the measure of E and by $U(x, \delta)$ - the neighborhood of x with radius $\delta > 0$. The quantity

$$D(E, x) = \lim_{\delta \to 0^+} \frac{|E \cap U(x, \delta)|}{2\delta}$$

if it exists, is the metric density of E at the point x. If $D(E, x) = 1$, the point x is called a point of density of E. It is well known that $D(E, x) = 1$ for almost all point x in E, i.e. if $D(E) = \{x \in E : D(E, x) = 1\}$, then $|D(E)| = |E|$. Note that if x is a point of density of E then the set $F = E \cap U(x, \delta)$ have positive measure. If $y \in (x - \delta, x + \delta)$ we identify y with e^{iy}, when working on the circle T.

Theorem 3.2. If ψ is a finite Blaschke product then

$$\left(\psi A(\bar{D}), L^\infty(D)\right)_b \subset \left(H^\infty(D) \cap W(D)\right) + C(\bar{D}) + V.$$

Proof: Let $f \in \left(\psi A(\bar{D}), L^\infty(D)\right)_b$. By theorem 3.1 f has non-tangential limits almost everywhere on T, $f^* \in \left(\psi A(T), L^\infty(T)\right)_b$ and $f^* \in W(D)$. As in ([3], theorem 2) we obtain $f^* \in H^\infty(T) + C(T)$. Thus $f^* \in H^\infty(D) + C(\bar{D})$. It follows that the function f^* belongs to the set $\left(H^\infty(D) \cap W(D)\right) + C(\bar{D})$. But $\left(H^\infty(D) \cap W(D)\right) + C(\bar{D}) + V \subset \left(\psi A(\bar{D}), L^\infty(D)\right)_b$ ([8]) and we obtain $g = f - f^* \in \left(\psi A(\bar{D}), L^\infty(D)\right)_b$.

Assume that $g \notin V$. Then there exists a sequence $\{r_n\}_n \subset (0,1)$, $r_n \to 1$ and $\delta > 0$ such that $\|g\chi_{D\setminus r_nD}(z)\| \geq \delta/2$ for all n. Let $A_n = \{z \in D : |g\chi_{D\setminus r_nD}(z)| > \delta/2\}$. As in $[2]$ we can find an interpolating sequence $\{z_n\}_n$, where z_n is a point of density of the set A_n for every n.
By lemma 2.1 $($ for $\delta = 1/2)$ there exists a subsequence $\{z_{n_j}\}_j$ of $\{z_n\}_n$ and a sequence $\{f_j\}_j$ in $\psi A(\bar{D})$ such that $f_j \to 0$ weakly in $\psi A(\bar{D})$, $\|f_j\| \leq 2$ and $\left|f_j(z_{n_j})\right| \geq \frac{1}{2}$ for all j.

For each n, by the continuity of f_j at z_{n_j}, there is a positive $\delta_j < 1 - |z_{n_j}|$ such that

$$\left|f_j(z_{n_j})\right| - |f_j(z)| \leq \left|f_j(z) - f_j(z_{n_j})\right| < \frac{1}{4},$$

whenever $|z - z_{n_j}| < \delta_j$. Therefore for every $z : |z - z_{n_j}| < \delta_j$ we obtain $|f_j(z)| > \frac{1}{4}$. Because z_{n_j} is a point of density of the set A_{n_j}, then the set

$$B_{n_j} = A_{n_j} \cap \{z \in \mathbb{C} : |z - z_{n_j}| < \delta_j\}$$

have positive measure and $|f_j(z)| > \frac{1}{4}$ for all $z \in B_{n_j}$. If $z \in B_{n_j}$ we have the inequality $|f_j(z)g(z)| > \frac{1}{4}.\frac{\delta}{2} = \delta/8$. Thus $\|f_jg\|_\infty \geq \delta/8$, because B_{n_j} have positive measure.

On the other hand, since $f_j \to 0$ weakly in $\psi A(\bar{D})$ and $g \in (\psi A(\bar{D}), L^\infty(D))_b$, there exists a sequence $\{g_j\}_j \subset \psi A(\bar{D})$ for which $\|f_j.g^* - g_j\|_\infty \leq \|f_j.g - g_j\|_\infty \to 0$. Because $g^* = 0$ almost everywhere on T we have $\|g_j\|_\infty \to 0$. Since for every $z \in D$

$$|f_j(z)g(z)| = |f_j(z)||g(z)| \leq 2\|g\|_\infty$$

then $\|f_jg\| \leq 2\|g\|_\infty$, i.e. $\|f_jg\|_\infty \to 0$.

This contradiction shows that $g \in V$ therefore $f = f^* + g \in H^\infty(D) + C(\bar{D}) + V$.

□

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